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ON k-CLOSURE OPERATORS IN GRAPHS. II

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0. Introduction. In this paper we study some problems which arose in connection with the notion of k-closure defined in [2]. Therefore, we recall first some notation and theorems from [2]. We shall use the terminology of [1].

By a ***graph*** we mean a couple $G = (U; X)$, where U is a non-empty set called the ***set of vertices or points*** and X is a family of 2-element subsets of U called the ***set of edges***. The edges will be written in the form $[ab]$ rather than in $\{a, b\}$.

We say that the vertices a, b are ***adjacent*** (which we shall denote by $a \leftrightarrow b$) if $[ab] \in X$.

Let $G = (U; X)$ be a graph and let $A \subseteq U$.

We say that a vertex $c \in U$ is ***k-reachable*** ($k > 0$) from the set A if there exist k different vertices $a_1, \ldots, a_k \in A$ such that $a_i \leftrightarrow c$ for $i = 1, 2, \ldots, k$. If c is k-reachable from A we shall write $c \underset{k}{\leftrightarrow} A$.

We say that a set A is ***k-closed*** in G if A contains all vertices k-reachable from A.

For $A \subseteq U$ let us denote by $C_k(A)$ the smallest k-closed set containing A.

Let $G = (U; X)$ be a graph, where $|U| = a > 1$; we say that the graph G is ***k-generated***, $1 \leqslant k < \min(\aleph_0, a)$, if for any $A \subseteq U$ such that $|A| = k$ we have $C_k(A) = U$.

We say that $G = (U; X)$ is an ***edge-minimal k-generated graph*** if G is k-generated and each graph $G' = (U; X')$, where $X' \subset X$, is not k-generated.

The function φ was defined in [2] as follows: for any two positive integers k and n ($1 \leqslant k < n$)

$$\varphi(n, k) = kn - \binom{k+1}{2}.$$

In [2], Theorem 4, it was proved that for any positive integer k and any cardinal a ($k < a$) there exists an edge-minimal k-generated graph $G_a^k = (U; X)$, where $|U| = a$ and

$$|X| = \begin{cases} a & \text{if } a \geqslant \aleph_0, \\ \varphi(n, k) & \text{if } a = n < \aleph_0. \end{cases}$$

The graph $G_n^k = (U; X)$ was defined as follows: $n = |U|$, $k = |U_0|$, where U_0 is a subset of U, $X = X_0 \cup X_1$, where X_0 consists of all 2-element subsets of U_0 and X_1 consists of all 2-element subsets $[uv]$, $u \in U_0$ and $v \in U \setminus U_0$.

The number $\varphi(n, k)$ is equal to the smallest integer m for which there exists a k-generated graph $G = (U; X)$ with $|U| = n$, $|X| = m$, $1 \leqslant k < n$.

Not every edge-minimal k-generated graph with n vertices ($1 \leqslant k < n$) has $\varphi(n, k)$ edges. In [2] an example of an edge-minimal 2-connected graph G containing 6 vertices and $\varphi(6, 2)+1$ edges was given. In this connection J. Płonka asked the following question:

Does there exist for any natural numbers k and n ($1 \leqslant k < n$) an edge-minimal k-generated graph $G = (U; X)$ such that $|U| = n$ and $|X| > \varphi(n, k)$?

In Section 1 we answer the question in the affirmative for $k > 1$ and $n \geqslant k+4$ (Theorem 1). The number $k+4$ ($k > 1$) is equal to the minimum of those natural numbers n for which there exists a graph $G = (U; X)$ with the required properties (Theorem 2).

In Section 2 we give an answer to the question of M. Sysło who asked what relation is between k-generation and k-connectivity. Namely, we prove that if $G = (U; X)$, where $|U| = n$, is k-generated ($1 \leqslant k < n$), then G is k-connected (Theorem 4). The converse is not true since (see Theorem 5) for any $k > 0$ there exists a 1-generated graph G the vertex-connectivity $\varkappa(G)$ of which is equal to k.

1. Edge-minimal k-generated graphs.

THEOREM 1. *For natural numbers k and n such that $k > 1$ and $n \geqslant k+4$ there exists an edge-minimal k-generated graph $G = (U; X)$ satisfying*

$$(*) \qquad |U| = n \quad \text{and} \quad |X| > \varphi(n, k).$$

Proof. Let $n = k+s$, $k > 1$ and $s \geqslant 4$. We shall consider two cases: (I) s is even and (II) s is odd.

(I) We define a graph $H_{k+s}^k = (U; X)$ as follows:

$$U = U_1 \cup U_2 \cup U_3,$$

where

$$U_1 = \{u_1, \ldots, u_{k-2}\}, \qquad U_2 = \{i, j\}, \qquad U_3 = \{w_1, \ldots, w_s\};$$

$$X = X_1 \cup X_2 \cup X_3,$$

where

$$X_1 = \{[uv]:\ u \in U_1, v \in U, v \neq u\}, \qquad X_2 = \{[iv] \text{ or } [jv]:\ v \in U_3\},$$
$$X_3 = \{[w_{2t+1}w_{2t+2}]:\ t = 0, \ldots, (s-2)/2\}.$$

We prove that the graph H^k_{k+s} satisfies (*). Observe that

$$|X| = |X_1| + |X_2| + |X_3| = (k-2)(s+2) + \frac{(k-2)(k-3)}{2} + 2s + \frac{s}{2}$$
$$= \frac{k^2 - k + 2ks + s - 2}{2},$$

so

$$|X| - \varphi(k+s, k) = \frac{s-2}{2} > 0.$$

Hence (*) holds.

We show that H^k_{k+s} is k-generated. We have to prove that for any $A \subseteq U$ such that $|A| = k$ we have $U \subseteq C_k(A)$. There are three cases: 1° $i, j \in A$, 2° $i \notin A$ and $j \notin A$, 3° $i \in A$ and $j \notin A$.

1° Obviously, $U_1 \subseteq C_k(A)$, so $U_2 \cup U_1 \subseteq C_k(A)$, and since for every $u \in U_3$ we have $u \underset{k}{\leftrightarrow} U_1 \cup U_2$, we get $U_3 \subseteq C_k(A)$. Thus $U \subseteq C_k(A)$.

2° Let $i \notin A$ and $j \notin A$. Then there exist $w_l, w_m \in U_3$ such that $w_l, w_m \in A$. Note that

$$i \underset{k}{\leftrightarrow} U_1 \cup \{w_l, w_m\} \quad \text{and} \quad j \underset{k}{\leftrightarrow} U_1 \cup \{w_l, w_m\}.$$

Thus by 1° we have $U \subseteq C_k(A)$.

3° Let $i \in A$ and $j \notin A$. Then there exists $w_l \in U_3 \cap A$. Hence

$$w_{l-1} \underset{k}{\leftrightarrow} U_1 \cup \{i, w_l\} \quad \text{or} \quad w_{l+1} \underset{k}{\leftrightarrow} U_1 \cup \{i, w_l\}.$$

Since $j \underset{k}{\leftrightarrow} U_1 \cup \{w_l, w_{l-1}\}$ and $j \underset{k}{\leftrightarrow} U_1 \cup \{w_l, w_{l+1}\}$, we have $j \in C_k(A)$. Further we argue as in 1°.

It remains to prove that the graph H^k_{k+s} is edge-minimal k-generated. We have already proved that H^k_{k+s} is k-generated, so it suffices to show that the graph $H^k_{k+s} - x$ is not edge-minimal, where x is an arbitrary edge of the graph H^k_{k+s}. For this purpose we consider 5 classes of edges:

(1) $P_1 = \{x = [uv]:\ u \in U_1, v \in U_1\}$,
(2) $P_2 = \{x = [uv]:\ u \in U_1, v \in U_2\}$,
(3) $P_3 = \{x = [uv]:\ u \in U_1, v \in U_3\}$,
(4) $P_4 = \{x = [uv]:\ u \in U_2, v \in U_3\}$,
(5) $P_5 = \{x = [uv]:\ u \in U_3, v \in U_3\}$.

One has to check that for any $x \in P_i$ $(i = 1, \ldots, 5)$ there exists a set A, where $|A| = k$ and $C_k(A) \subset U$. Table 1 gives some examples for every class P_i.

Table 1

i	x	A
1	$[u_{k-3}u_{k-2}]$	$\{u_1, \ldots, u_{k-3}, w_1, w_2, i\}$
2	$[u_{k-2}w_1]$	$\{u_1, \ldots, u_{k-3}, w_1, w_2, i\}$
3	$[u_{k-2}i]$	$\{u_1, \ldots, u_{k-3}, w_1, w_2, i\}$
4	$[iw_1]$	$\{u_1, \ldots, u_{k-2}, w_1, j\}$
5	$[w_1w_2]$	$\{u_1, \ldots, u_{k-2}, w_1, j\}$

(II) Let $n = k+s$, $k > 1$, $s \geqslant 4$ and let s be odd. We define the graph $H'^{k}_{k+s} = (U'; X')$ as follows:

$$U' = U, \quad X' = X_1' \cup X_2' \cup X_3',$$

where

$$X_1' = X, \quad X_2' = X_2 \setminus \{[w_s j]\},$$

$$X_3' = \left\{x = [w_{s-1}w_s] \text{ or } x = [w_{2t+1}w_{2t+2}]\colon t = 0, \ldots, \frac{s-3}{2}\right\}.$$

We have

$$|X'| = \frac{k^2 - k + 2ks + s - 3}{2},$$

so (*) holds. The proof that H' is edge-minimal k-generated is similar to that of case (I).

Observe that the assumption $n \geqslant k+4$ in Theorem 1 is essential, since we have

THEOREM 2. *The number $k+4$ ($k > 1$) is equal to the minimum of those natural numbers n for which there exists an edge-minimal k-generated graph $G = (U; X)$ satisfying* (*).

Proof. The existence of the graph G for the number $n = k+4$ follows from Theorem 1. We prove that for $n < k+4$ such a graph G does not exist.

Let K_n denote the complete graph with n vertices, i.e., a graph without loops any two different vertices of which are adjacent.

Let $e(K_n)$ denote the number of edges of K_n, i.e.,

$$e(K_n) = \frac{n(n-1)}{2}.$$

The values of $q(n) = e(K_n) - \varphi(n, k)$ for $k < n < k+4$ are $q(k+1) = 0$, $q(k+2) = 1$, $q(k+3) = 3$. It is obvious that in the first two cases such a graph G does not exist. Consider the third case. Since $q(k+3) = 3$, the graph G can have $e(K_n) - 2$, $e(K_n) - 1$ or $e(K_n)$ edges. We prove that if $|X| = e(K_n) - 2$, then G is k-generated but not edge-minimal. Let

$A \subseteq U$ and $|A| = k$; then there exists $c_1 \notin A$ such that $c_1 \underset{k}{\leftrightarrow} A$. Further, there exists $c_2 \notin A \cup \{c_1\}$ such that $c_2 \underset{k}{\leftrightarrow} A \cup \{c_1\}$ and there exists $c_3 \notin A \cup \{c_1, c_2\}$ such that $c_3 \underset{k}{\leftrightarrow} A \cup \{c_1, c_2\}$. Thus G is k-generated.

To prove that G is not edge-minimal it suffices to consider two cases:

1° $G = (U; X)$, where $U = U_1 \cup U_2$, and

$$U_1 = \{u_1, \ldots, u_{k-1}\}, \quad U_2 = \{w_1, w_2, w_3, w_4\}, \quad [w_1 w_2] \notin X, \ [w_3 w_4] \notin X.$$

2° $G = (U; X)$, where $U = U_1 \cup U_2$, and

$$U_1 = \{u_1, \ldots, u_k\}, \quad U_2 = \{w_1, w_2, w_3\}, \quad [w_1 w_2] \notin X, \ [w_1 w_3] \notin X.$$

1° We prove that the graph $G' = (U'; X')$, where $U = U'$ and $X' = X \setminus \{[w_2 w_3]\}$, is k-generated. There are three cases:

(a) $w_2, w_3 \in A$; then $w_4 \underset{k}{\leftrightarrow} U_1 \cup \{w_2\}$. Since $U_1 \subset C_k(A)$, we have $U' \subseteq C_k(A)$.

(b) $w_2 \notin A$, $w_3 \notin A$; then $w_1 \in A$ or $w_4 \in A$. It is easy to see that in both cases we get $U \subseteq C_k(A)$.

(c) $w_2 \notin A$, $w_3 \in A$; then

$$w_1 \underset{k}{\leftrightarrow} U_1 \cup \{w_3\}, \quad w_4 \underset{k}{\leftrightarrow} U_1 \cup \{w_1, w_3\} \quad \text{and} \quad w_2 \underset{k}{\leftrightarrow} U_1 \cup \{w_1, w_3, w_4\}.$$

2° The graph $G'' = (U''; X'')$, where $U'' = U$ and $X'' = X \setminus \{[w_2 w_3]\}$, is k-generated, since for any $u \in U$ we get $u \underset{k}{\leftrightarrow} U_1$ and for any $A \subseteq U$ we have $U_1 \subseteq C_k(A)$.

We have proved that if G has $e(K_n) - 2$ edges, then G is k-generated but not edge-minimal. It is obvious that if $|X| = e(K_n) - 1$ or $|X| = e(K_n)$, then the same holds.

When we consider the k-generation or edge-minimality of a graph, it is sometimes convenient to study properties of the complement of the graph. Let $G = (U; X)$ be a graph. Write $\overline{X} = \{[ab]\colon a, b \in U, \ [ab] \notin X\}$. The complement of the graph G is $\overline{G} = (U; \overline{X})$.

LEMMA 1. *Let $G = (U; X)$ with $|X| > k$ and $\overline{G} = (U; \overline{X})$ satisfy the following condition*:

(C) *there exists $A \subseteq U$, where $|A| = k$, and for any $u \in U \setminus A$ there exists $w \in A$ such that $[uw] \in \overline{X}$.*

Then G is not k-generated.

Proof. Let $\overline{G} = (U; \overline{X})$ satisfy (C); then for any $u \in U \setminus A$ there exists $w \in A$ such that $[uw] \notin X$. Thus $C_k(A) = A$ and G is not k-generated.

Let $G = (U; X)$, where $U = \{u_1, \ldots, u_n\}$. Let $\pi(\overline{G}) = (c_1, \ldots, c_n)$, where c_i denotes the degree of the vertex u_i in the graph $\overline{G}$ ($i \in \{1, \ldots, n\}$).

LEMMA 2. *If $G = (U; X)$ is an edge-minimal k-generated graph, then*

(i) $c_i \leqslant (n-1)-k$ *for* $i \in \{1, \ldots, n\}$;

(ii) *in* $\pi(\overline{G})$ *there are at most k elements equal to* 0;

(iii) *if in* $\pi(\overline{G})$ *there are exactly k elements equal to* 0, *then G is isomorphic to the graph G_n^k defined in Section* 0;

(iv) *if $n = k+s$ and there are $k-1$ zeros in* $\pi(\overline{G})$, *then*

$$\sum_{i=1}^{n} c_i = s(s-1).$$

Proof. (i) If G is k-generated, then for any vertex u_i we have $\deg u_i \geqslant k$.

(ii) If in $\pi(\overline{G})$ there are more than k elements equal to 0, then G is not edge-minimal.

(iii) We have

$$\pi(\overline{G_n^k}) = (\underbrace{0, \ldots, 0}_{k}, \underbrace{n-1-k, \ldots, n-1-k}_{s}).$$

Let

$$\pi(\overline{G}) = (\underbrace{0, \ldots, 0}_{k}, c_{k+1}, \ldots, c_n),$$

where $c_{k+1}, \ldots, c_n$ are different from 0. Since G is an edge-minimal k-generated graph, we infer that if $[u_j u_i] \in X$ and $k+1 \leqslant j \leqslant n$, then $1 \leqslant i \leqslant k$. Thus G is isomorphic to G_n^k.

(iv) Let $n = k+s$ and take $\pi(\overline{G}) = (0, \ldots, 0, c_k, \ldots, c_n)$. Let $H = (U_0, Y)$ be the subgraph of the graph G induced by $U_0 = \{u_k, \ldots, u_n\}$. Since in G there are $k-1$ vertices adjacent to any vertex of G, the subgraph H is 1-generated, so connected. Since G is edge-minimal, H does not contain cycles. So H is a tree with $s+1$ vertices and s edges. Thus the graph $\overline{H}$ has $e(K_{s+1})-s$ edges. Consequently, $\pi(\overline{G}) = (0, \ldots, 0, c_k, \ldots, c_n)$ satisfies

$$\sum_{i=1}^{n} c_i = 2\big(e(K_{s+1})-s\big) = (s+1)s-2s = s(s-1).$$

THEOREM 3. *For $n = k+4$ $(k > 1)$ there exists a unique (up to isomorphism) edge-minimal k-generated graph $G = (U; X)$ satisfying* $(*)$.

Proof. We are interested only in graphs with n vertices and q edges, where $n = k+4$ and $e(K_n)-q < e(K_n)-\varphi(n, k) = 6$. It will be convenient to consider complements of such graphs. Consider all possible sequences $\pi(\overline{G})$ for which $|U| = k+4$ $(k > 1)$ and $|\overline{X}| = e(K_{k+4})-q < 6$. By Lemma 2 we can restrict the number of sequences $\pi(\overline{G})$. Using Lemma 1 one can check that if there are less than $k-2$ zeros in $\pi(\bar{G})$, then G is not k-generated. It remains to consider only a few sequences $\pi(\bar{G})$. The results are contained in Table 2. We use the following notation: $G = (U; X)$, $\bar{G} = (U; \bar{X})$, $U = U_1 \cup U_2$, $U_1 = \{u_1, \ldots, u_{k-2}\}$, $U_2 = \{w_1, \ldots, w_6\}$. If $u \in U_1$, then $\deg u = n-1$.

Table 2

$\pi(\bar{G})$	$\bar{X}$	Description of G
$(\underbrace{0, \ldots, 0}_{k-2}, 1, 1, 1, 1, 3, 3)$	$[w_6w_5], [w_6w_1], [w_6w_2], [w_5w_3], [w_5w_4]$	not k-generated
$(\underbrace{0, \ldots, 0}_{k-2}, 1, 1, 1, 2, 2, 3)$	$[w_6w_5], [w_6w_4], [w_6w_3], [w_1w_2], [w_4w_5]$	not k-generated
	$[w_6w_5], [w_6w_4], [w_6w_3], [w_1w_4], [w_2w_5]$	not edge-minimal
	$[w_6w_1], [w_6w_2], [w_6w_4], [w_5w_3], [w_5w_4]$	not k-generated
$(\underbrace{0, \ldots, 0}_{k-2}, 1, 1, 2, 2, 2, 2)$	$[w_1w_2], [w_3w_4], [w_3w_5], [w_4w_6], [w_1w_6]$	edge-minimal k-generated
	$[w_1w_3], [w_2w_3], [w_4w_5], [w_4w_6], [w_5w_6]$ $[w_1w_3], [w_2w_4], [w_3w_5], [w_4w_6], [w_5w_6]$	not k-generated
$(\underbrace{0, \ldots, 0}_{k-2}, 1, 1, 1, 1, 1, 3)$	$[w_1w_6], [w_2w_6], [w_3w_6], [w_4w_5]$	not k-generated
$(\underbrace{0, \ldots, 0}_{k-2}, 1, 1, 1, 1, 2, 2)$	$[w_1w_5], [w_2w_5], [w_3w_6], [w_4w_6]$	not k-generated
	$[w_1w_2], [w_3w_5], [w_4w_6], [w_5w_6]$	not edge-minimal
$(\underbrace{0, \ldots, 0}_{k-2}, 1, 1, 1, 1, 1, 1)$	$[w_1w_2], [w_3w_4], [w_5w_6]$	not edge-minimal

2. Connectivity of k-generated graphs. We give now some relations between k-generated graphs and k-connected graphs. First observe some properties of k-connected graphs.

LEMMA 3. *Let $G = (U; X)$ be k-connected and $v \notin U$. Let $G^* = (U \cup \{v\}; X \cup X_1)$ be a new graph, where $X_1 = \{[vu_1], [vu_2], \ldots, [vu_s]\}$, $s \geqslant k$, and $u_1, u_2, \ldots, u_s$ are different elements of U. Then G^* is k-connected.*

Proof. Let $\varkappa(G)$ denote the vertex-connectivity of G. Thus $\varkappa(G) \geqslant k$, so the smallest number of vertices splitting G, i.e., making G not connected or trivial, is not less than k. We show that the same holds for G^*. Let S be the set of vertices splitting G^*. We have three possibilities:

(1) $v \in S$,

(2) $v \notin S$ and $\{w\colon w \leftrightarrow v\} \subseteq S$,

(3) $v \notin S$ and $\{w\colon w \leftrightarrow v\} \nsubseteq S$.

(1) Since $v \in S$, we have $|S| \geqslant \varkappa(G)+1$. Moreover, after removing the vertex v, we get the graph G.

(2) In this case we have $|S| \geqslant \deg v$.

(3) $|S| \geqslant \varkappa(G)$ because using the vertex v we get new chains connecting vertices of the graph G.

In any of these three cases we get $\varkappa(G^*) \geqslant \varkappa(G)$, hence G^* is k-connected.

LEMMA 4. *Any k-generated graph $G = (U; X)$ for which $|U| = n$, $1 \leqslant k < n$, contains a subgraph K_{k+1}.*

Proof. By Lemma 2 in [2] it is known that G contains a subgraph $K_k = (U_0, X_0)$. Since $C_k(U_0) = U$, there exists $a \in U \setminus U_0$ such that $a \underset{k}{\leftrightarrow} U_0$. Thus any two different elements in the set $U_0 \cup \{a\}$ are adjacent.

THEOREM 4. *Let $G = (U; X)$ with $|U| = n$ be k-generated* $(1 \leqslant k < n)$. *Then G is k-connected.*

Proof. Put $n = k+s$. We use induction on s. Lemma 4 gives the first step. The inductive step follows from Lemma 3.

The k-connectivity of a graph does not imply the k-generation of it. However, we have

THEOREM 5. *For any integers $k > 0$ and $n \geqslant 2k$ there exists a 1-generated graph $G = (U; X)$ such that $\varkappa(G) = k$, $|U| = n$, and G is not 2-generated.*

Proof. Put $U = \{v_1, \ldots, v_k, u_1, \ldots, u_{n-k}\}$, $X = Y \setminus (X_1 \cup X_2)$, where Y is the set of all 2-element subsets of U, $X_1 = \{[v_1 v_i]: i = 2, \ldots, k\}$, $X_2 = \{[u_1 u_j]: j = 2, \ldots, n-k\}$.

It is easy to show that G is k-connected and only 1-generated.

LEMMA 5. *If $G = (U; X)$ is a graph such that $\varkappa(G) = k$, $|U| = n$ and $k < n < 2k$, then G is s-generated for each $s < n/(n-k)$.*

Proof. For $v \in U$ and $W \subset U$ we write

$$\deg(v, W) = |\{w \in W: w \leftrightarrow v\}|, \qquad \overline{\deg}(v, W) = |\{w \in W: \operatorname{non}(w \leftrightarrow v)\}|.$$

For $A \subset U$ and $s = |A|$ let

$$A_1 = \{v: v \underset{s}{\leftrightarrow} A\}, \qquad A_2 = \{v: v \underset{s}{\leftrightarrow} A \cup A_1\} \setminus (A \cup A_1).$$

Since $\varkappa(G) = k$, we have $\overline{\deg}(v, U \setminus \{v\}) \leqslant n-k-1$ for any $v \in U$. Hence

$$|\{v: \operatorname{non}(v \underset{s}{\leftrightarrow} A)\} \setminus A| \leqslant |A|(n-k-1).$$

Observe that the condition $s < n/(n-k)$ is equivalent to $s(n-k-1) < n-s$, which means that $A_1 \neq \emptyset$. Consider the set $U \setminus (A \cup A_1 \cup A_2)$. Put $|U \setminus (A \cup A_1 \cup A_2)| = r$. If $u \in U \setminus (A \cup A_1 \cup A_2)$, then $\overline{\deg}(u, A \cup A_1) \geqslant |A_1|+1$. Consequently, we get

$$\begin{aligned} r(|A_1|+1) &\leqslant (n-k-1)|A \cup A_1| - |A_2| \\ &\leqslant (n-k-1)(s+|A_1|) - (n-s-|A_1|-r), \end{aligned}$$

whence

$$r < \frac{(n-k)\big(n/(n-k)+|A_1|\big)-n}{|A_1|} < n-k,$$

which implies $|A \cup A_1 \cup A_2| > k$.

We shall prove that for any $v \in U \setminus (A \cup A_1 \cup A_2)$ we have $v \overset{s}{\leftrightarrow} A \cup A_1 \cup A_2$. Otherwise, there exists $u_0 \in U \setminus (A \cup A_1 \cup A_2)$ such that

$$\overline{\deg}(u_0, A \cup A_1 \cup A_2) > k-(s-1).$$

Then

$$\overline{\deg}(u_0, A \cup A_1 \cup A_2) > k-s+1 > k - \frac{n}{n-k} + 1 = \frac{k}{n-k}(n-k-1) > n-k-1,$$

which contradicts the assumption $\varkappa(G) = k$.

LEMMA 6. *The inequality $s < n/(n-k)$ in Lemma 5 cannot be strengthened.*

Proof. We construct a graph $G = (U; X)$ such that $|U| = n$, $\varkappa(G) = k$, $k < n < 2k$ and G is not t-generated for $t = \{n/(n-k)\}$ (1).

Put

$$U = \{u_1, \ldots, u_t, v_1, \ldots, v_{n-t}\}.$$

We split the set U into the classes

$$K = \{u_1, \ldots, u_t\},\ K_1 = \{v_1, \ldots, v_d\},\ K_2 = \{v_{d+1}, \ldots, v_{2d}\}, \ldots,$$
$$K_i = \{v_{(i-1)d+1}, \ldots, v_{id}\}, \ldots, K_q = \{v_{(q-1)d+1}, \ldots, v_{n-t}\},$$

where $d = n-k-1$. Let $X = Y \setminus X_1$, where Y is the set of all 2-element subsets of U and $X_1 = \{[u_i v_j]\colon 1 \leqslant i \leqslant q,\ v_j \in K_i\}$.

If $n = k+1$, then $t = n$ and G is not a t-generated graph.

If $n > k+1$, then $t < n$ and $K_1 \cup \ldots \cup K_q \neq \emptyset$. We also have

$$t \geqslant \frac{n}{n-k}, \quad t \geqslant \frac{n-t}{n-k-1}, \quad \left\{\frac{n-t}{n-k-1}\right\} = q,$$

whence $t \geqslant q$.

Since for any v_j there exists $u \in K$ such that $\mathrm{non}(v_j \leftrightarrow u)$, so by Lemma 1 the graph G is not t-generated.

Given a graph G, let us denote by $g(G)$ the maximum of all numbers s for which G is s-generated.

THEOREM 6. *If a graph $G = (U; X)$ is such that $|U| = n$ and $\varkappa(G) = k$, then $g(G) \geqslant \{n/(n-k)\} - 1$ and this inequality cannot be strengthened.*

(1) $\{m\} = -[-m]$ (see [1]).

Proof. From the definition of k-connectivity it follows that $n > k$.

If $n \geqslant 2k$, then by $\varkappa(G) \geqslant 1$ the graph G is 1-connected, and so 1-generated. Thus $g(G) \geqslant 1 = \{n/(n-k)\}-1$. The inequality is sharp by Theorem 5.

If $k < n < 2k$, our theorem follows from Lemmas 5 and 6.

REFERENCES

[1] F. Harary, *Graph theory*, Reading, Mass., 1969.

[2] J. Płonka, *On k-closure operators in graphs*, Colloquium Mathematicum 43 (1980), p. 373-381.

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