

NOTE ON QUASI-UNIFORM SPACES AND REPRESENTABLE SPACES

BY

P. FLETCHER (BLACKSBURG, VIRGINIA)

1. Introduction. In this paper we define the concept of a representable topological space. With each representable space we associate a compatible quasi-uniformity in a natural way. Using this quasi-uniformity we show that a connected representable space is homogeneous and that the full homeomorphism group of a representable space is a topological semigroup under the topology of quasi-uniform convergence.

A general introduction to quasi-uniform spaces may be found in [5].

Throughout this paper \circ denotes the usual composition of relations.

2. Preliminaries. Let X be a non-empty set and let \mathcal{U} be a filter on $X \times X$ such that

- (i) each element of \mathcal{U} is a reflexive relation on X ,
- (ii) if $U \in \mathcal{U}$, there exists $W \in \mathcal{U}$ such that $W \circ W \subset U$.

Then \mathcal{U} is a *quasi-uniformity on X* .

Let X be a set and let \mathcal{U} be a quasi-uniformity on X . Let $\mathcal{T}_{\mathcal{U}} = \{A \subset X: \text{if } a \in A \text{ then there exists } U \in \mathcal{U} \text{ such that } U(a) \subset A\}$. Then $\mathcal{T}_{\mathcal{U}}$ is a topology on X , called the *quasi-uniform topology on X generated by \mathcal{U}* . If (X, \mathcal{T}) is a topological space and \mathcal{U} is a quasi-uniformity on X such that $\mathcal{T} = \mathcal{T}_{\mathcal{U}}$, then \mathcal{U} is a *compatible quasi-uniformity*.

We let $H(X)$ denote the group of all homeomorphisms from a space X onto itself and let i denote the identity of $H(X)$. If $A \subset X$, then $A' = \{h \in H(X): h|_A = i|_A\}$ and if G is a subgroup of $H(X)$, then $G' = \{x \in X: g(x) = x \text{ for each } g \in G\}$. If $G \subset H(X)$ and $A \subset X$, then $G(A) = \{g(a): g \in G, a \in A\}$. We often write A'' for $(A')'$, x' for $\{x\}'$ and $G(x)$ for $G(\{x\})$.

Definition. A topological space (X, \mathcal{T}) is *representable* provided that if F is a closed set and $x \in X - F$, then $F'(x)$ is a neighborhood of x .

3. Representable spaces. We prove

THEOREM 1. *Let (X, \mathcal{T}) be a representable space and for each $A \subset X$ let $U_A = (\bigcup \{\{x\} \times A'(x): x \in X - A\}) \cup (A \times X)$. Let $\beta = \{U_A: A \text{ is closed}\}$. Then β is a subbase for a compatible quasi-uniformity \mathcal{U} on X .*

Proof. For each $A \subset X$, $\Delta \subset U_A = U_A \circ U_A$ so that β is a subbase for a quasi-uniformity \mathcal{U} on X . Let $x \in A \in \mathcal{T}$. Then $U_{X-A}(x) = (X-A)'(x) \subset A$. Thus $\mathcal{T} \subset \mathcal{T}_{\mathcal{U}}$. Now let A be a proper $\mathcal{T}_{\mathcal{U}}$ -open set such that $x \in A$. Then there is a finite collection $\{F_i\}_{i=1}^n$ of closed subsets of X such that

$$x \notin \bigcup_{i=1}^n F_i \quad \text{and} \quad x \in \bigcap_{i=1}^n [U_{F_i}(x)] \subset A.$$

Since

$$x \notin \bigcup_{i=1}^n F_i, \quad \bigcap_{i=1}^n [U_{F_i}(x)] = \bigcap_{i=1}^n F'_i(x)$$

and since (X, \mathcal{T}) is a representable space, $\bigcap_{i=1}^n F'_i(x)$ contains a \mathcal{T} -open set about x . Thus $\mathcal{T} \subset \mathcal{T}_{\mathcal{U}}$ and \mathcal{U} is a compatible quasi-uniformity.

COROLLARY. *Let (X, \mathcal{T}) be a representable space, let F be a closed set and let $x \in X - F$. Then $F'(x) \in \mathcal{T}$.*

Proof. Let $z \in U_F(x)$. Then $U_F(z) \subset U_F \circ U_F(x) = U_F(x)$. Thus $F'(x) = U_F(x) \in \mathcal{T}_{\mathcal{U}} = \mathcal{T}$.

THEOREM 2. *Every connected representable space is homogeneous.*

Proof. Let (X, \mathcal{T}) be a representable space. Let $U_{\varphi} = \bigcup \{\{x\} \times \times H(X)(x) : x \in X\}$. Let $x \in X$. For each $y \in X - U_{\varphi}(x)$, $U_{\varphi}(x) \cap U_{\varphi}(y) = \emptyset$.

It follows from the Corollary to Theorem 1, that $U_{\varphi}(x)$ is both open and closed. Hence $H(X)(x) = U_{\varphi}(x) = X$.

Since the entourages of the subbase β of Theorem 1 are reflexive and transitive, β is not a subbase for a uniformity whenever (X, \mathcal{T}) is not 0-dimensional ([1], Theorem 1). Nevertheless, if for each $U \in \mathcal{U}$ we let

$$W(U) = \{(f, g) \in H(X) \times H(X) : (f(x), g(x)) \in U \text{ for each } x \in X\}$$

and let

$$\mathcal{W} = \{W(U) : U \in \mathcal{U}\},$$

then \mathcal{W} is a quasi-uniformity on $H(X)$. The quasi-uniformity \mathcal{W} is called the *quasi-uniformity of quasi-uniform convergence with respect to \mathcal{U}* and $\mathcal{T}_{\mathcal{W}}$ is the *topology of quasi-uniform convergence for $H(X)$* . The topology of quasi-uniform convergence has been studied in [6].

THEOREM 3. *Let (X, \mathcal{T}) be a representable homogeneous Hausdorff space, let \mathcal{U} be the compatible quasi-uniformity for X described in Theorem 1, and let $H(X)$ have the topology of quasi-uniform convergence with respect to \mathcal{U} . Then $H(X)$ is a topological semigroup.*

Proof. By Theorem 1 of [2], it suffices to show that if $h \in H(X)$, then h is a \mathcal{U} -quasi-uniformly continuous function. Let $h \in H(X)$ and let A be a closed subset of X so that $U_A \in \mathcal{U}$. Then $U_{h^{-1}(A)} \in \mathcal{U}$. Let $(x, y) \in U_{h^{-1}(A)}$.

If $h(x) \in A$, then $(h(x), h(y)) \in U_A$. If $h(x) \notin A$, then $x \notin h^{-1}(A)$ so that $y \in (h^{-1}(A))'(x)$. There exists $g \in (h^{-1}(A))'$ so that $g(x) = y$. Then $h \circ g \circ h^{-1} \in A'$ and $h \circ g \circ h^{-1}(h(x)) = h(y)$. It follows that $(h(x), h(y)) \in U_A$.

Definition [4]. A space X is S.L.H. (*strong local homogeneity*) if for every neighborhood of any point x , there exists a subneighborhood $U(x)$ such that for any $z \in U(x)$ there exists a homeomorphism $g \in (X - U(x))'$ with $g(x) = z$.

Definition [3]. A space X is a *Galois space* provided that for each closed set F , $F = F''$.

THEOREM 4. *Every S.L.H. space without isolated points is a representable (Galois) space.*

Proof. Let X be an S.L.H. space without isolated points, let F be a closed subset of X and let $x \in X - F$. There is a neighborhood U of x such that $U \subset (X - F) \cap (X - U)'(x)$. Then $F \subset X - U$ so that $x \in U \subset (X - U)(x) \subset F'(x)$. Thus X is representable.

Clearly $F \subset F''$. Suppose that $y \in F'' \cap (X - F)$. Then $\{y\} = F'(y)$ is an open set, since X is representable. This contradicts the assumption that X has no isolated points. Consequently, $F = F''$ and X is a Galois space.

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