COLLOQUIUM MATHEMATICUM

VOL. XXXVIII 1977 FASC. 1

ON THE ARITY OF AFFINE MODULES

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According to [4], for a given algebra $\mathfrak{A} = (A; \boldsymbol{F}), \boldsymbol{A}(\boldsymbol{F})$ and $\boldsymbol{A}^{(n)}(\boldsymbol{F})$ will denote the family of all algebraic operations and the family of all n-ary algebraic operations, respectively. (We note that in [1] Grätzer uses the term "polynomial" instead of "algebraic operation".) We shall not distinguish algebras with the same set of algebraic operations. An n-ary operation $f(x_1, \ldots, x_n)$ of $\mathfrak{A}$ is called *idempotent* if $f(x, \ldots, x) = x$ holds for each $x \in A$. The *idempotent reduct* of the algebra $\mathfrak{A} = (A; \boldsymbol{F})$ is defined to be the algebra $\mathscr{I}(\mathfrak{A}) = (A; \boldsymbol{I}(\boldsymbol{F}))$ with $\boldsymbol{I}(\boldsymbol{F})$ the family of all idempotent algebraic operations of $\mathfrak{A}$. The *arity* of an algebra $(A; \boldsymbol{F})$ was defined by E. Marczewski as the minimal natural number n with the property

$$(A; \boldsymbol{F}) = (A; \boldsymbol{A}^{(n)}(\boldsymbol{F})).$$

In this note by a *ring* we always mean an associative ring with unit element, written 1. The ideal of a ring R generated by the set $\{r(1-r) \mid r \in R\}$ will be called the *Booleanizer* of R and will be denoted by R^*. For a ring R a *right* (*left*) *affine R-module* is defined to be the idempotent reduct of a unitary right (left) R-module. Unless otherwise stated, we deal with right affine modules. Obviously, the operations of an affine R-module are of the form

$$x_1 r_1 + \ldots + x_n r_n \quad \text{with } r_i \in R \text{ and } \sum_{i=1}^{n} r_i = 1.$$

The family of these operations will be denoted by $\boldsymbol{R}$. If the affine R-module is the idempotent reduct of the module $\mathfrak{M}$, then the annihilator ideal of $\mathfrak{M}$ will be called the *annihilator ideal of the affine module*. The affine R-module will be called *trivial* if its annihilator ideal is R.

The aim of this paper is to determine the arity of affine modules. Our theorem extends the result of Płonka [4] who investigating the arity of the idempotent reduct of (not necessarily Abelian) groups proved that, restricting to Abelian groups, this arity equals two if and only if the group is of odd exponent, else it equals three.

In [3] Ostermann and Schmidt showed that, for any ring R and any affine R-module $(A; \boldsymbol{R})$,

$$(A; \boldsymbol{R}) = \left(A; \boldsymbol{A}^{(2)}(\boldsymbol{R}) \cup \{f\}\right), \quad \text{where } f(x, y, z) = x + y(-1) + z.$$

This implies evidently that the arity of $(A; \boldsymbol{R})$ equals two if and only if $f \in \boldsymbol{A}\left(\boldsymbol{A}^{(2)}(\boldsymbol{R})\right)$, else it is equal to three. For brevity we set $\boldsymbol{B} = \boldsymbol{A}\left(\boldsymbol{A}^{(2)}(\boldsymbol{R})\right)$.

THEOREM. *Let R be a ring. The arity of a non-trivial affine R-module $(A; \boldsymbol{R})$ with annihilator ideal I equals two if and only if*

$$R = I + R^*. \tag{1}$$

Proof. First we note that it suffices to prove the Theorem for $I = 0$. Indeed, if I is non-trivial, then $(A; \boldsymbol{R})$ can be regarded as an affine R/I-module with operations defined by

$$x_1\bar{r}_1 + \ldots + x_n\bar{r}_n = x_1 r_1 + \ldots + x_n r_n,$$

$$r_i \in R, \bar{r}_i = r_i + I, \ i = 1, 2, \ldots, n, \ \sum_{i=1}^{n} r_i = 1.$$

Since this affine module has a trivial annihilator ideal, it is of arity two if and only if $R/I = (R/I)^*$, which by the equality $(R/I)^* = (R^* + I)/I$ is equivalent to (1).

Assume now that $I = 0$. Observe that the equality of operations

$$x_1 r_1 + \ldots + x_n r_n = x_1 s_1 + \ldots + x_n s_n, \quad r_i, s_i \in R, \sum_{i=1}^{n} r_i = \sum_{i=1}^{n} s_i = 1,$$

implies the equality of their coefficients, i. e., $r_i = s_i$ for all $i = 1, 2, \ldots, n$.

Let us introduce the following notation:

$$J = \{r \mid r \in R, x + y(-r) + zr \in \boldsymbol{B}\}.$$

We shall show that $J = R^*$ which, by the observation made above, implies that $f \in \boldsymbol{B}$ is equivalent to $1 \in J$, and hence to $R = R^*$, as required.

Since for any $s, t \in J$ and $r \in R$ we have

$$x + y(-s+t) + z(s-t) = \left(x + y(-s) + zs\right) + z(-t) + yt,$$

$$x + y(-sr) + zsr = \left(x + y(-s) + zs\right)r + x(1-r),$$

$$x + y(-rs) + zrs = x + \left(yr + x(1-r)\right)(-s) + \left(zr + x(1-r)\right)s,$$

J is an ideal of R. Thus the inclusion $J \supseteq R^*$ follows from the equality

$$x + yr(r-1) + zr(1-r) = \left(x(1-r) + zr\right)(1-r) + \left(x(2-r) + y(r-1)\right)r$$

implying that this operation is contained in $\boldsymbol{B}$.

To show $J \subseteq R^*$, observe that

$$\{rr'(1-r) \mid r, r' \in R\} \subseteq R^*. \tag{2}$$

(In fact, this set generates R^*.) Indeed, if $r, r' \in R$, then

$$rr'(1-r) = (rr'-r'r)(1-r)+r'r(1-r)$$

belongs to R^*, since R/R^* is a Boolean ring, and hence it is commutative. It suffices to prove that for any ternary operation

$$x_1r_1+x_2r_2+x_3r_3 \in \boldsymbol{B}, \quad r_i \in R,\ i = 1,2,3,\ r_1+r_2+r_3 = 1,$$

and for any $u \in R$ we have

$$r_iur_j \in R^*, \quad i,j = 1,2,3, \tag{3}$$

provided $i \neq j$. Indeed, if $r \in J$, i.e., $x+y(-r)+zr \in \boldsymbol{B}$, then choosing $u = 1$ we have $1 \cdot r \in R^*$, as desired.

By (2), our claim in (3) is obvious for operations contained in $A^{(2)}(\boldsymbol{R})$. Assume now that our claim holds for the operations

$$x_1s_1+x_2s_2+x_3s_3,\ x_1t_1+x_2t_2+x_3t_3 \in \boldsymbol{B},$$

$$s_i, t_i \in R,\ i = 1, 2, 3,\ \sum_{i=1}^{3} s_i = \sum_{i=1}^{3} t_i = 1,$$

and prove it for

$$x_1r_1+x_2r_2+x_3r_3 = (x_1s_1+x_2s_2+x_3s_3)r+(x_1t_1+x_2t_2+x_3t_3)(1-r),$$

where r is any element of R. By a simple computation we have

$$\begin{aligned} r_iur_j &= \big(s_ir+t_i(1-r)\big)u\big(s_jr+t_j(1-r)\big) \\ &= s_irus_jr+t_i(1-r)us_jr+s_irut_j(1-r)+t_i(1-r)ut_j(1-r). \end{aligned}$$

Thus we can apply our assumption to the first and fourth terms, and inclusion (2) to the second and third terms to conclude that $r_iur_j \in R^*$ provided $u \in R$ and $i \neq j$. This completes the proof of the Theorem.

Let $\mathscr{C}$ denote the class of those rings R for which the arity of each non-trivial affine R-module equals two. An immediate consequence of the Theorem is

COROLLARY 1. *$R \in \mathscr{C}$ if and only if R coincides with its Booleanizer.*

This clearly implies that the class $\mathscr{C}$ does not depend on whether we consider left or right affine modules. An obvious consequence of Corollary 1 is the following characterization of the class $\mathscr{C}$:

COROLLARY 2. *$R \in \mathscr{C}$ if and only if the prime field F_2 of characteristic 2 is not a homomorphic image of R.*

Finally, using this characterization we prove

COROLLARY 3. *The class consisting of the zero ring and of all rings not belonging to $\mathscr{C}$ forms a quasivariety.*

Proof. Let us denote this class by $\overline{\mathscr{C}}$. In order to prove that $\overline{\mathscr{C}}$ is a quasivariety we can apply Mal'cev's preservation theorem stating that an algebraic class containing the one-element algebra and closed under prime products, direct products and subalgebras is a quasivariety (see [2], p. 271).

The notation in this proof is from [1]. We note that the zero element and the unit element of the rings are regarded to be elements distinguished by nullary operations.

Assume that $(R_i \mid i \in I)$ is a non-void family of rings in $\overline{\mathscr{C}}$ and that $\mathscr{D}$ is a prime filter over I. If

$$Q = \{i \mid i \in I, R_i \text{ is the zero ring}\} \in \mathscr{D},$$

then the prime product $\Pi_{\mathscr{D}}(R_i|i \in I)$ is the zero ring. In the opposite case, $\mathscr{D}' = \{D|D \subseteq I', D \in \mathscr{D}\}$ is prime over $I' = I - Q$ and

$$\Pi_{\mathscr{D}}(R_i|i \in I) \cong \Pi_{\mathscr{D}'}(R_i|i \in I').$$

Since F_2 is a homomorphic image of R_i for all $i \in I'$, $(F_2)^{I'}_{\mathscr{D}'}$ is a homomorphic image of $\Pi_{\mathscr{D}'}(R_i|i \in I')$. Moreover, $(F_2)^{I'}_{\mathscr{D}'}$ is a Boolean ring with $0 \neq 1$, hence F_2 is its homomorphic image. This proves

$$\Pi_{\mathscr{D}}(R_i|i \in I) \in \overline{\mathscr{C}}.$$

By a similar but much simpler argument one can show that $\overline{\mathscr{C}}$ is closed under direct products and subrings. The proof of Corollary 3 is complete.

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Reçu par la Rédaction le 5. 8. 1975