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ON SOME PROPERTIES OF POST ALGEBRAS WITH COUNTABLE CHAIN OF CONSTANTS

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Post algebras play the same role in many-valued logics as Boolean algebras do in the classical logic. Viewed as algebraical systems with $n \geqslant 2$ constants, Post algebras were introduced in 1942 by P. C. Rosenbloom. In papers of C. C. Chang, Ph. Dwinger, G. Epstein, H. Rasiowa, G. Rousseau, T. Traczyk and E. Włodarska such algebras were later examined or applied to metalogical problems of m-valued logics.

The class of algebras which are coproducts of a Boolean algebra and of a lattice fulfilling some additional conditions — as introduced by Dwinger (see [1]) — is a generalization of the concept of Post algebras of any finite order $n \geqslant 2$.

Generalized Post algebras which are coproducts of a Boolean algebra and of the interval $\langle 0, 1\rangle$ were characterized by Traczyk (see [8]) as distributive lattices with the greatest and the least elements, and with a set of zero-argument and unary operations satisfying a system of axioms. Such a characterization is possible for any Post algebra which is a coproduct of a Boolean algebra and of a linearly ordered set T with the least and the greatest elements. Analogous to the definition of Post algebras of order $n \geqslant 2$, introduced by Traczyk (see [6]), such a characterization allows to transfer certain theorems on Post algebras of a finite order to the case of generalized Post algebras.

The purpose of this paper is to present a generalization of the Loomis-Sikorski theorem and of the Rasiowa-Sikorski lemma which were proved for Post algebras of any finite order by T. Traczyk and E. Włodarska, respectively, to the case of Post algebras with a countable chain of constants.

1. Characterizations of generalized Post algebras. Let P be a distributive lattice with zero and unit elements $(\wedge, \vee)$ and let B be the Boolean algebra consisting of all complemented elements of P.

Definition 1.1. P is a *generalized Post algebra* of type T, where T is an arbitrary linearly ordered set with the greatest element 1 and the least element 0, if and only if there exists an indexed set $\{e_t\}_{t\in T}$ of elements in P and an indexed set $\{D_t\}_{t\in T}$ of unary operations on P such that the following conditions are satisfied:

P1. $e_0 = \wedge$; $e_1 = \vee$; for arbitrary $t, t' \in T$, if $t \leqslant t'$, then $e_t \leqslant e_{t'}$.

P2. $D_0(x) = \vee$; $D_t(x) \in B$ for every $t \in T$; for arbitrary $t, t' \in T$, if $t \leqslant t'$, then $D_t(x) \geqslant D_{t'}(x)$.

P3. $x = (P) \bigcup_{t\in T} D_t(x) \cap e_t$ for every $x \in P$ (1).

P4. If, for every $t \in T$, $a_t \in B$, $a_0 = \vee$ and $a_t \geqslant a_{t'}$ for $t \leqslant t'$, then there exists the join $x = (P) \bigcup_{t\in T} a_t \cap e_t$ and $a_t = D_t(x)$ for each $t \in T$.

The algebra defined in this way will be denoted by $P = \langle \{e_t\}_{t\in T}; B\rangle$.

It can be easily checked that P is a sublattice of the Cartesian product of an indexed set $\{B_t\}_{t\in T}$ consisting of copies of the Boolean algebra B (for each $t \in T$, $B_t = B$) (for a proof, see [8]).

LEMMA 1.1. *Let P satisfy conditions* P1-P4 *and let $x, y \in P$. The following statements hold true*:

(a) $D_t(x) \cup D_t(y) = D_t(x \cup y)$.

(b) $D_t(x) \cap D_t(y) = D_t(x \cap y)$.

(c) *$x \leqslant y$ if and only if $D_t(x) \leqslant D_t(y)$ for each $t \in T$.*

(d) *$D_t(a) = a$ for every $a \in B$ and for every $t > 0$.*

(e) *$D_t(e_{t'}) = \vee$ for $t \leqslant t'$ and $D_t(e_{t'}) = \wedge$ for $t > t'$.*

(f) *If $x_i \in B$ for every $i \in I$, then the join $(P) \bigcup_{i\in I} x_i$ (the meet $(P) \bigcap_{i\in I} x_i$) exists if and only if it does exist in B. If they do exist, they are equal.*

(g) *If $x_i \in P$ for every $i \in I$, then the join $(P) \bigcup_{i\in I} x_i$ (the meet $(P) \bigcap_{i\in I} x_i$) exists if and only if the join $(P) \bigcup_{i\in I} D_t(x_i)$ (the meet $(P) \bigcap_{i\in I} D_t(x_i)$) does exist for every $t \in T$. If it exists, then $D_t\big((P) \bigcup_{i\in I} x_i\big) = (P) \bigcup_{i\in I} D_t(x_i)$ $\big(D_t\big((P) \bigcap_{i\in I} x_i\big) = (P) \bigcap_{i\in I} D_t(x_i)\big)$.*

The proof is analogous to that in [8].

Definition 1.2. P is a *P_0-lattice* of type T, where T is an arbitrary linearly ordered set with the greatest element 1 and the least element 0, if and only if there exists an indexed set $\{e_t\}_{t\in T}$ of elements in P such that:

Q1. $e_0 = \wedge$; $e_1 = \vee$; for arbitrary $t, t' \in T$, if $t \leqslant t'$, then $e_t \leqslant e_{t'}$.

Q2. $x = (P) \bigcup_{t\in T} x_t \cap e_t$ for every $x \in P$, where the indexed set $\{x_t\}_{t\in T}$ satisfies the condition

(a) $x_0 = \vee$; $x_t \in B$ for every $t \in T$; for arbitrary $t, t' \in T$, if $t \leqslant t'$, then $x_t \geqslant x_{t'}$.

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(1) If $\{x_t\}_{t\in T}$ is an indexed set of elements in a Post algebra P (in a Boolean algebra B), then the symbols $(P) \bigcup_{t\in T} x_t$ $((B) \bigcup_{t\in T} x_t)$ and $(P) \bigcap_{t\in T} x_t$ $((B) \bigcap_{t\in T} x_t)$ will denote the join and the meet of all x_t for $t \in T$ in P (in B), respectively.

Q3. If an indexed set $\{x_t\}_{t\in T}$ satisfies condition (a), then there exists the join $x = (P) \bigcup_{t\in T} x_t \cap e_t$.

LEMMA 1.2. *Let P be a P_0-lattice of type T and let an indexed set $\{x_t\}_{t\in T}$ satisfy condition* (a) *in* Q2. *Then there is $\big((P) \bigcup_{t\in T} x_t e_t\big) \cap e_{t'} = (P) \bigcup_{t\leqslant t'} x_t e_t$ for every $t' \in T$* (²).

Proof. Let us denote $(P) \bigcup_{t\leqslant t'} x_t e_t$ by N and $\big((P) \bigcup_{t\in T} x_t e_t\big) \cap e_{t'}$ by M. Since $N = (P) \bigcup_{t\leqslant t'} x_t e_t \leqslant (P) \bigcup_{t\in T} x_t e_t$ and $N = (P) \bigcup_{t\leqslant t'} x_t e_t \leqslant (P) \bigcup_{t\leqslant t'} e_t = e_{t'}$, we have $N \leqslant \big((P) \bigcup_{t\in T} x_t e_t\big) \cap e_{t'} = M$.

On the other hand, since $\{x_t\}_{t\in T}$ satisfies condition (a) in Q2, we obtain $(P) \bigcup_{t\in T} x_t e_t \leqslant (P) \bigcup_{t\leqslant t'} x_t e_t \cup x_{t'}$. Hence we have $M = \big((P) \bigcup_{t\in T} x_t e_t\big) \cap e_{t'} \leqslant (P) \bigcup_{t\leqslant t'} x_t e_t \cup x_{t'} e_{t'} = (P) \bigcup_{t\leqslant t'} x_t e_t$, which completes the proof.

Definition 1.3. Let P be a P_0-lattice of type T. If an indexed set $\{x_t\}_{t\in T}$ of elements in P satisfies condition (a) in Q2 and $x = (P) \bigcup_{t\in T} x_t e_t$ for some $x \in P$, then $\{x_t\}_{t\in T}$ is called a *representation* of the element x in P.

THEOREM 1.1. *P_0-lattice of type T is a generalized Post algebra of type T if and only if it satisfies the condition*

Q4. *If $a \in B$ and $a \cap e_{t'} \leqslant (P) \bigcup_{t<t'} e_t$ for some $t' \in T$, then $a = \wedge$.*

The proof of this theorem is preceded by the following

LEMMA 1.3. *If P satisfies conditions* Q1-Q4, *then the representation of each element in P is unique, that is, if $\{x_t\}_{t\in T}$ and $\{y_t\}_{t\in T}$ are representations of the same element, then $x_t = y_t$ for each $t \in T$.*

Proof. Suppose that there exist indexed sets $\{x_t\}_{t\in T}$ and $\{y_t\}_{t\in T}$ which satisfy condition (a) in Q2 and are such that

$$x = (P) \bigcup_{t\in T} x_t e_t \quad \text{and} \quad x = (P) \bigcup_{t\in T} y_t e_t.$$

Let $x_{t'} \neq y_{t'}$ for some $t' \in T$. Since $x_{t'}$ and $y_{t'}$ belong to B, there exist in P elements $-x_{t'}$ and $-y_{t'}$. In virtue of lemma 1.2,

$$x \cap e_{t'} = (P) \bigcup_{t\leqslant t'} x_t e_t = (P) \bigcup_{t\leqslant t'} y_t e_t,$$

whence

$$x_{t'} e_{t'} \leqslant (P) \bigcup_{t\leqslant t'} y_t e_t.$$

Thus

$$\begin{aligned}(-y_{t'}) x_{t'} e_{t'} \leqslant \big((P) \bigcup_{t\leqslant t'} y_t e_t\big) \cap (-y_{t'}) &= \big((P) \bigcup_{t<t'} y_t e_t\big) \cap (-y_{t'}) \\ &\leqslant (P) \bigcup_{t<t'} y_t e_t \leqslant (P) \bigcup_{t<t'} e_t.\end{aligned}$$

(²) xy denotes the meet of x and y as well as $x \cap y$.

Since P satisfies condition Q4, $(-y_{t'})x_{t'} = \wedge$. Replacing $x_{t'}$ by $y_{t'}$, we get $(-x_{t'})y_{t'} = \wedge$. Thus $x_{t'} = y_{t'}$, q.e.d.

Proof of the theorem. Let P be a P_0-lattice of type T and let P satisfy condition Q4. We shall show that P satisfies conditions P1-P4. Condition P1 is obvious. Define an indexed set of unary operations on P by setting $D_t(x) = x_t$, where $\{x_t\}_{t\in T}$ is a representation of an element x. According to lemma 1.3, this definition is correct and conditions P2-P4 are trivially satisfied.

Now let P be a generalized Post algebra of type T. P is evidently a P_0-lattice of type T and we shall show that P satisfies condition Q4. Suppose that $ae_{t'} \leqslant (P)\bigcup_{t<t'} e_t$ for some $a \in B$ and some $t' \in T$. Using lemma 1.1, we obtain

$$D_{t'}(ae_{t'}) = D_{t'}(a)\cap D_{t'}(e_{t'}) = a \leqslant D_{t'}\big((P)\bigcup_{t<t'} e_t\big) = (P)\bigcup_{t<t'} D_{t'}(e_t) = \wedge,$$

whence $a = \wedge$, q.e.d.

2. Representation theorem. Let $P = \langle\{e_t\}_{t\in T};\ B\rangle$ be an arbitrary (but from now on fixed) Post algebra of type T. An ideal Δ in P is said to be of *order* t' if and only if $(P)\bigcup_{t<t'} e_t \in \Delta$ and $e_{t'} \notin \Delta$.

Let X_0 denote a set of all prime ideals in the Boolean algebra B and let X_t denote the set of all prime ideals of order t in P for each $t\in T$. We define the mapping $\Phi_t\colon X_t \to X_0$ for every $t \in T$ by the formula $\Phi_t(\Delta) = \Delta \cap B$. It is easy to verify that

(a) Φ_t is a one-to-one mapping and its range is X_0,

(b) for each $t\in T$, if $\Delta\in X_t$, then $x\in\Delta$ if and only if $D_t(x)\in \Delta\cap B$.

Now, for each $a\in B$, put $h_0(a) = \{\Delta\in X_0\colon a\notin\Delta\}$, $h(a) = \bigcup_{t\in T}\Phi_t^{-1}\big(h_0(a)\big)$, $F_0 = \{h_0(a)\colon a\in B\}$, $F = \{h(a)\colon a\in B\}$ and $X = \bigcup_{t\in T} X_t$.

The set F defined by the last but one formula is a field of subsets of the set X, and h is an isomorphism of the Boolean algebra B onto F. If we consider the set X as a topological space, with the set F as a subbasis, then X is a compact Hausdorff space, F is the class of all both closed and open sets in X and X_t is a dense subset of X for each $t\in T$.

The topological space X is called the *Stone space* for the Post algebra P.

Let $E_0 = \emptyset$, $E_{t'} = \bigcup_{t\leqslant t'} X_t$ for $t'\in T$ and let R be the P_0-lattice of sets determined by an indexed set $\{E_t\}_{t\in T}$ and by the Boolean field F. Then R is a Post field of sets of type T isomorphic with the Post algebra P.

The proof is similar to that in the case of Post algebras of finite order.

3. The Rasiowa - Sikorski lemma for generalized Post algebras. Let $P = \langle\{e_t\}_{t\in T};\ B\rangle$ be a generalized Post algebra of type T, where $\overline{\overline{T}} \leqslant \aleph_0$.

Let (Q) be an enumerable set of infinite joins and meets in P,

$$(Q) = \{x_k:\ x_k = (P)\bigcup_{l\in L'_k} x_{k,l}\}_{k\in N} \cup \{y_k:\ y_k = (P)\bigcap_{l\in L''_k} y_{k,l}\}_{k\in N},$$

and let $(D_t Q)$ for $t\in T$ be the set of infinite joins and meets in B,

$$(D_t Q) = \{a_k^t:\ a_k^t = (B)\bigcup_{l\in L'_k} a_{k,l}^t\}_{k\in N} \cup \{b_k^t:\ b_k^t = (B)\bigcap_{l\in L''_k} b_{k,l}^t\}_{k\in N},$$

where N is the set of positive integers, L'_k and L''_k are arbitrary sets and, for all $t\in T$, $k\in N$, we have $a_k^t = D_t(x_k)$, $b_k^t = D_t(y_k)$, $a_{k,l}^t = D_t(x_{k,l})$ and $b_{k,l}^t = D_t(y_{k,l})$.

Definitions of Q-ideals and D_tQ-ideals are the same as in [9].

The existence of Q-ideals of order t for each $t\in T$ in algebra P follows from the existence of D_tQ-ideals for all $t\in T$ in the Boolean algebra B.

Let $X_0^{D_tQ}$ for every $t\in T$ denote the set of all D_tQ-ideals in B and let X_t^Q denote the set of all Q-ideals of order t in P. Put $X_Q = \bigcup_{t\in T} X_t^Q$. Then X_Q is a dense subset of the Stone space X for the Post algebra P.

Let $B(X_Q)$ be the class of all sets of the form $\bigcup_{t\in T} \Phi_t^{-1}(U)\cap X_t^Q$ for all $U\in F_0$. The class $B(X_Q)$ is a field of sets.

Now let R_Q be the class of all subsets of X_Q of the form $\bigcup_{t\in T} A_t E_t^Q$, where $E_0^Q = \emptyset$ and $E_{t'}^Q = \bigcup_{t\leqslant t'} X_t^Q$ for each $t'\in T$. Let an indexed set $\{A_t\}_{t\in T}$ satisfies the condition

(a) $A_0 = X_Q$; $A_t\in B(X_Q)$ for every $t\in T$; for arbitrary t, t', if $t\leqslant t'$, then $A_t\geqslant A_{t'}$.

Then R_Q is a generalized Post field of sets of type T.

Define a mapping h_Q^+ from the Post algebra P into Post field R_Q by the formula

$$h_Q^+(x) = \bigcup_{t\in T} h_Q\big(D_t(x)\big)\cap E_t^Q,$$

where $h_Q(a) = h(a)\cap X_Q$ is an isomorphism from the Boolean algebra B into the Boolean field of sets $B(X_Q)$. Then h_Q^+ is a Q-isomorphism (i.e., an isomorphism preserving joins and meets in (Q)) from the generalized Post algebra P of type T into the generalized Post field of sets R_Q of type T. This proves the following

THEOREM 3.1. *For every enumerable set (Q) of infinite joins and meets in a generalized Post algebra of type T, where $\overline{\overline{T}} \leqslant \aleph_0$, there is a Q-isomorphism from this algebra into a generalized Post field of sets of type T.*

Proofs of all facts in this section are analogous to those for Post algebras with the finite chain of constants.

4. The Loomis - Sikorski theorem for some generalized Post algebras. Let $\mathfrak{M}$ be a fixed infinite cardinal number.

Definition 4.1. A subset H of a topological space X is said to be *$\mathfrak{M}$-closed* if it is the intersection of at most $\mathfrak{M}$ sets, both closed and open in X.

Definition 4.2. A subset H of a topological space X is said to be *$\mathfrak{M}$-nowhere dense* if it is a subset of a nowhere dense $\mathfrak{M}$-closed set.

Definition 4.3. A subset H of a topological space X is said to be of the *$\mathfrak{M}$-category* if it is the union of at most $\mathfrak{M}$ sets $\mathfrak{M}$-nowhere dense in X.

Definition 4.4. A Boolean $\mathfrak{M}$-algebra (i.e., an $\mathfrak{M}$-complete algebra) is called *$\mathfrak{M}$-representable* if it is isomorphic to a quotient algebra F/Δ, where F is an $\mathfrak{M}$-field of sets and Δ is an $\mathfrak{M}$-ideal of F.

Definition 4.5. A generalized Post algebra of type T is *$\mathfrak{M}$-representable* if it is isomorphic to a quotient Post algebra R_m/Δ_m, where $R_m = \langle\{E_t\}_{t\in T};\ F_m\rangle$ is a P_0-lattice of sets of type T, F_m is a Boolean $\mathfrak{M}$-field, and Δ_m is an $\mathfrak{M}$-ideal of sets in F_m.

Let $P = \langle\{e_t\}_{t\in T};\ B\rangle$ be a fixed generalized Post $\mathfrak{M}$-algebra of type T, where $\overline{\overline{T}} \leqslant \mathfrak{M}$, let $R = \langle\{E_t\}_{t\in T};\ F\rangle$ be a generalized Post field of sets of type T, isomorphic to P, and let h be an isomorphism from P onto R. F_m denotes the least $\mathfrak{M}$-field of subsets of Stone space X containing F, and Δ_m denotes the $\mathfrak{M}$-ideal of all sets in F_m of the $\mathfrak{M}$-category.

LEMMA 4.1. *F_m coincides with the class of all subsets of X of the form $H\cup N\setminus N'$, where $H\in F$ and $N, N'\in \Delta_m$.*

LEMMA 4.2. *Let a Boolean algebra B be $\mathfrak{M}$-representable. If $A\cap E_{t'}\subset \bigcup_{t<t'} E_t$ for some $A\in F_m$ and some $t'\in T$, then $A\in\Delta_m$.*

Proofs of lemmas 4.1 and 4.2 are similar to those in [7].

Now let us denote by R_m a P_0-lattice of sets of type T determined by an indexed set $\{E_t\}_{t\in T}$ and by the Boolean field of sets F_m, $R_m = \langle\{E_t\}_{t\in T};\ F_m\rangle$.

We define in R_m a relation $\sim$ in the following way: for $A = \bigcup_{t\in T} A_tE_t$ and $A' = \bigcup_{t\in T} A'_tE_t$, there is $A\sim A'$ if and only if $A_t\setminus A'_t\cup A'_t\setminus A_t\in\Delta_m$ for every $t\in T$.

LEMMA 4.3. *If a Boolean algebra B is $\mathfrak{M}$-representable,*

$$A = \bigcup_{t\in T} A_tE_t \quad \textit{and} \quad A = \bigcup_{t\in T} A'_tE_t,$$

then

$$\bigcup_{t\in T} A_tE_t \sim \bigcup_{t\in T} A'_tE_t.$$

Proof. By lemma 1.2 we have

$$AE_{t'} = \bigcup_{t\leqslant t'} A_tE_t = \bigcup_{t\leqslant t'} A'_tE_t,$$

whence

$$(-A'_{t'})A_{t'}E_{t'} \leqslant (-A'_{t'}) \cap \bigcup_{t \leqslant t'} A'_t E_t \leqslant \bigcup_{t<t'} A'_t E_t \leqslant \bigcup_{t<t'} E_t.$$

And by lemma 4.2, $(-A'_{t'})A_{t'} \in \Delta_m$. The proof that $(-A_{t'})A'_{t'} \in \Delta_m$ is analogous. Consequently, $A'_{t'} \setminus A_{t'} \cup A_{t'} \setminus A'_{t'} \in \Delta_m$, q.e.d.

In the next lemmas we assume that B is an $\mathfrak{M}$-representable Boolean algebra.

LEMMA 4.4. *The relation $\sim$ is an equivalence and it is a congruence with respect to the operations $\cup$ and $\cap$.*

The easy proof is omitted.

By R_m/Δ_m we denote the set of all residue classes of the relation $\sim$. It is easy to verify that R_m/Δ_m is a distributive lattice with zero and unit elements under the following operations:

$$[X] \cup [Y] = [X \cup Y] \quad \text{and} \quad [X] \cap [Y] = [X \cap Y].$$

LEMMA 4.5. *Let K be an arbitrary set of the cardinality $\overline{\overline{K}} \leqslant \mathfrak{M}$ and let*

$$A = \bigcup_{t \in T} A_t E_t \quad \textit{and} \quad A_k = \bigcup_{t \in T} A_{k,t} E_t$$

for $k \in K$ be arbitrary elements of R_m. If

$$A = \bigcup_{k \in K} A_k \quad (A = \bigcap_{k \in K} A_k),$$

then

$$[A] = \bigcup_{k \in K} [A_k] \quad ([A] = \bigcap_{k \in K} [A_k]).$$

Proof. It is easy to verify that $[A] \leqslant [B]$ if and only if $A_t \setminus B_t \in \Delta_m$ or every $t \in T$. Assume that $[A_k] \leqslant [S]$ for each $k \in K$, where $S = \bigcup_{t \in T} S_t E_t$. Hence $A_{k,t} \setminus S_t \in \Delta_m$ for every $k \in K$ and $t \in T$. Since Δ_m is an $\mathfrak{M}$-ideal, it follows that $\bigcup_{t \in T} A_{k,t} \setminus S_t \in \Delta_m$ for each $t \in T$. Thus, in virtue of lemma 4.3,

$$[A] = [\bigcup_{t \in T} A_t E_t] = [\bigcup_{t \in T} \bigcup_{k \in K} A_{k,t} E_t] \leqslant [S], \quad \text{q.e.d.}$$

LEMMA 4.6. *$R_m/\Delta_m = \langle \{[E_t]\}_{t \in T}; F_m/\Delta_m \rangle$ is a generalized Post algebra of type T.*

Proof. Since R_m/Δ_m is a distributive lattice with zero $[\emptyset]$ and unit $[X]$ elements, it is sufficient to show that R_m/Δ_m satisfies conditions Q1-Q4.

Q1 is obvious.

Q2. Let $[A] \in R_m/\Delta_m$. Since $A \in R_m$, we have $A = \bigcup_{t \in T} A_t E_t$, where an indexed set $\{A_t\}_{t \in T}$ satisfies the condition

(a) $A_0 = X$; $A_t \in F_m$ for each $t \in T$; for arbitrary $t, t' \in T$, if $t \leqslant t'$, then $A_t \geqslant A_{t'}$.

In virtue of lemma 4.5,

$$[A] = \bigcup_{t\in T} [A_t][E_t],$$

where $\{[A_t]\}_{t\in T}$ satisfies analogous condition as $\{A_t\}_{t\in T}$. Therefore, R_m/Δ_m satisfies condition Q2.

Q3. Let an indexed set $\{[A_t]\}_{t\in T}$ satisfy condition (a). We shall show that there exists an element $A \in R_m$ such that

$$[A] = \bigcup_{t\in T} [A_t][E_t].$$

Since F_m is an $\mathfrak{M}$-field of sets, there exists a meet $\bigcap_{t\leqslant t'} A_t = S_{t'}$ for each $t' \in T$. In virtue of lemma 4.5,

$$[S_{t'}] = [\bigcap_{t\leqslant t'} A_t] = \bigcap_{t\leqslant t'} [A_t] = [A_{t'}].$$

Now, since $\{S_t\}_{t\in T}$ satisfies condition (a) in Q2, there exists the union $A = \bigcup_{t\in T} S_t E_t$ in R_m. Using again lemma 4.5, we infer that

$$[A] = \bigcup_{t\in T} [S_t][E_t] = \bigcup_{t\in T} [A_t][E_t], \quad \text{q.e.d.}$$

Q4. Let $[A] \in F_m/\Delta_m$ and $[AE_{t'}] \leqslant \bigcup_{t<t'} [E_t]$ for some $t' \in T$. We shall show that then $[A] = [\emptyset]$. Using lemma 5, we get $\bigcup_{t<t'} [E_t] = [\bigcup_{t<t'} E_t]$. Since $[AE_{t'}] \leqslant [\bigcup_{t<t'} E_t]$, we have

$$[AE_{t'} \cup \bigcup_{t<t'} E_t] = [\bigcup_{t<t'} E_t], \qquad AE_{t'} \cup \bigcup_{t<t'} E_t = \bigcup_{t\in T} B_t E_t,$$

where $B_t = X$ for $t < t'$, $B_{t'} = A$ and $B_t = \emptyset$ for other $t \in T$. According to the definition of the relation $(\sim)$, if $\bigcup_{t\in T} B_t E_t \in [\bigcup_{t<t'} E_t]$, then $B_{t'} \setminus \emptyset \in \Delta_m$. Since $B_{t'} = A$, consequently $A \in \Delta_m$ and $[A] = [\emptyset]$, q.e.d.

THEOREM 4.1. *The generalized Post $\mathfrak{M}$-algebra $P = \langle \{e_t\}_{t\in T}; B\rangle$ of type T is $\mathfrak{M}$-representable if and only if the Boolean algebra B is $\mathfrak{M}$-representable.*

Proof. Since the algebra P is $\mathfrak{M}$-complete and $\mathfrak{M}$-representable, so, by the definition, is B. To prove the sufficiency we define a mapping $\hat{h}$ by the formula $\hat{h}(a) = [h_0(a)]$ for $a \in B$. It is easy to verify that $\hat{h}$ is an $\mathfrak{M}$-isomorphism and that $\hat{h}$ maps B onto F_m/Δ_m. The isomorphism $\hat{h}$ may be extended to an isomorphism h^* from algebra P into R_m/Δ_m.

Let

$$x_k = \bigcup_{t\in T} D_t(x_k) e_t$$

belong to P for $k \in K$, where $\overline{\overline{K}} \leqslant \mathfrak{M}$, and let

$$h^*(x_k) = \bigcup_{t\in T} \hat{h}\big(D_t(x_k)\big) \cap [E_t].$$

We shall show that h^* is an $\mathfrak{M}$-isomorphism. Let $x = \bigcap_{k\in K} x_k$. Using lemmas 1.1 and 4.5, we obtain the following chain of equations:

$$h^*(x) = h^*(\bigcap_{k\in K} x_k) = \bigcup_{t\in T} \hat{h}\big(D_t(\bigcap_{k\in K} x_k)\big) \cap [E_t] = \bigcup_{t\in T} \bigcap_{k\in K} \hat{h}\big(D_t(x_k)\big) \cap [E_t]$$
$$= \bigcap_{k\in K} \bigcup_{t\in T} \hat{h}\big(D_t(x_k)\big) \cap [E_t] = \bigcap_{k\in K} h^*(x_k).$$

In a similar way one can show that if $y = \bigcup_{k\in K} y_k$, where $y_k \in P$ and $\overline{\overline{K}} \leqslant \mathfrak{M}$ for each $k \in K$, then

$$h^*(y) = \bigcup_{k\in K} h^*(y_k).$$

Since by hypothesis $\hat{h}$ maps B onto F_m/Δ_m, it follows that h^* maps P onto R_m/Δ_m. This completes the proof.

COROLLARY. *Since every σ-complete Boolean algebra is σ-representable, every generalized σ-complete Post algebra of type T, where $\overline{\overline{T}} \leqslant \aleph_0$, is σ-representable.*

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