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VARIETIES OF MODULAR p-ALGEBRAS

BY

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A description of varieties or equational classes of distributive p-algebras was given in Lee [8]. In [6] we have investigated the varieties of modular S-algebras, which form a subclass of the class of all modular p-algebras. In this note we shall continue this research and study varieties of modular p-algebras associated with algebras B_m^n, closely related to distributive varieties.

In the second section we characterize in terms of identities all varieties generated by a (finite) set of algebras B_m^n. The third section deals with a description of the lattice of all those varieties. In both sections we get generalizations of corresponding results appearing in [8].

1. Preliminaries. A universal algebra $\langle L; \cup, \cap, {}^*, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ is called a *(modular) p-algebra* iff $\langle L; \cup, \cap, 0, 1\rangle$ is a bounded (modular) lattice and if, for every $a \in L$, the element a^* is a *pseudocomplement* of a, i.e., $x \leqslant a^*$ iff $a \cap x = 0$. The class of all (modular) p-algebras is equationally definable (see [1]), and is, therefore, a variety. Standard results on p-algebras can be found in [1].

For a p-algebra L define the set $B(L) = \{x \in L; x = x^{**}\}$ of closed elements. The partial ordering of L partially orders $B(L)$ and makes the latter into a Boolean algebra $\langle B(L); \vee, \cap, {}^*, 0, 1\rangle$ for which

$$a \vee b = (a \cup b)^{**}$$

holds. Another significant subset of a p-algebra L is the set of dense elements $D(L) = \{x \in L; x^* = 0\}$. $D(L)$ is a filter in L.

A modular p-algebra satisfies the identity

$$x = x^{**} \cap (x \cup x^*) \tag{1}$$

and, obviously, $x \cup x^* \in D(L)$ (see [1]). For $a \in B(L)$, write

$$F_a = \{x \in L; x^{**} = a\}.$$

Obviously, $F_0 = \{0\}$ and $F_1 = D(L)$.

An element x of a p-algebra L is said to be a *Stone element* (see [7]) if it satisfies the identity

$$x^* \cup x^{**} = 1.$$

The following is true (see 4.3 in [7]):

1.1. *In a modular p-algebra L, the subset*

$$S(L) = \{x \in L; x \text{ is a Stone element}\}$$

is a subalgebra of L.

Subdirectly irreducible modular p-algebras were characterized in Theorem 1 of [7] as follows:

1.2. *Let L be a modular p-algebra. Then L is subdirectly irreducible iff L satisfies the following conditions:*

(i) *$D(L)$ is subdirectly irreducible;*

(ii) *for each closed Stone element a $(0 < a < 1)$ of L, we have* $\mathrm{card}(F_a) \geqslant 2$.

We shall study varieties of p-algebras which are generated by subdirectly irreducible modular p-algebras L having $D(L) \simeq M_n$. (M_1 is the one-element lattice, and M_2 – the two-element one. For the cardinal number $n \geqslant 3$, let M_n denote the modular lattice of dimension 2 and order $n+2$.) In [5] we have given the full description of these p-algebras. The following proposition will be useful for our purposes (see 1.4 in [5]):

1.3. *Let L be a subdirectly irreducible modular p-algebra with $D(L) \simeq M_n$ $(n \geqslant 3)$. Then*

(i) $B(L) \cap S(L) = \{0, 1\}$ (*type* I);

(ii) $B(L) \cap S(L) = \{0, a, a^*, 1\}$ (*type* II).

We shall now construct some subdirectly irreducible modular p-algebras L with $D(L) \simeq M_n$. Let B be a Boolean lattice. Take the ordinal sum $B \oplus M_n$. Identifying the largest element of B with the smallest element of M_n, we get a bounded modular lattice B^n. Moreover, B^n is a pseudocomplemented lattice with $D(B^n) \simeq M_n$. The lattice B^n can be considered as a modular p-algebra. According to 1.2 and 1.3, B^n $(n \geqslant 2)$ is subdirectly irreducible of type I. (Evidently, $B^1 \simeq B$.) For the sake of brevity, let B_m denote the 2^m-element (m is a non-negative number) and B_ω an infinite Boolean algebra. (For $n = 1$, only B_1^1 is subdirectly irreducible.) Some of these algebras are pictured in Fig. 1.

Now we can make our task more precise. We shall investigate $V(B_{m_1}^{n_1}, \ldots, B_{m_r}^{n_r})$, the smallest variety of modular p-algebras generated by the finite set $\{B_{m_1}^{n_1}, \ldots, B_{m_r}^{n_r}\}$. For $n = 2$, B^n is distributive, and all varieties $V(B^2)$ are described in [8].

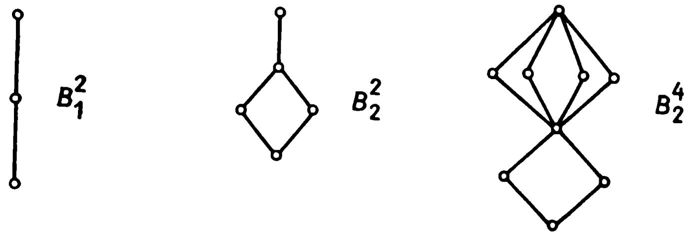

Fig. 1

2. Properties of $V(B_{m_1}^{n_1}, \ldots, B_{m_r}^{n_r})$.

2.1. *Every finitely generated subalgebra of a p-algebra B^n is finite.* The proof is straightforward.

From 2.1 it follows immediately

2.2. *If B and C are Boolean algebras, then the following statements are true:*

(i) $V(B^n) = V(C^m)$ *for B, C, n and m infinite;*

(ii) $V(B^n) = V(B^m)$ *for n and m infinite;*

(iii) $V(B^n) = V(C^n)$ *for B and C infinite.*

These results justify the notation $V(B_\omega^\omega)$ for $V(B_\omega^n)$, and $V(B_m^\omega)$ for $V(B_m^n)$, if n is infinite.

2.3. *Let $L \in V(B_m^n)$ for $1 \leqslant n$, $m \leqslant \omega$. Then L satisfies the following identities* ([1]):

(J) $$x_1 \cap [x_2 \cup (x_3 \cap x_4)] \cap (x_3 \cup x_4) \leqslant x_2 \cup (x_1 \cap x_3) \cup (x_1 \cap x_4);$$

(2) $$x \cap (y^* \cup z) = (x \cap y^*) \cup (x \cap z).$$

Moreover, if n ($n \geqslant 2$) or m are finite, then L satisfies one of the identities:

(J_n) $$y \cap \Big[\bigwedge_{1 \leqslant i < j \leqslant n} (x_i \cup x_j)\Big] \leqslant \bigvee_{1 \leqslant i \leqslant n} (y \cap x_i)$$

or

(L_m) $$(x_1 \cap \ldots \cap x_m)^* \cup (x_1^* \cap \ldots \cap x_m)^* \cup \ldots \cup (x_1 \cap \ldots \cap x_m^*)^* = 1.$$

Proof. It is enough to check conditions (J), (J_n), (L_m) and (2) for B_m^n. We know that $D(B_m^n) \simeq M_n$. By [4], M_n satisfies (J) and, for n finite also (J_n). Now it is easy to check (J) and (J_n) for B_m^n. (L_m) can be proved as in [8]. (2) follows from the fact that each sublattice of B_m^n generated by the set $\{x, y^*, z\}$ is distributive.

2.4. *Let L be a modular p-algebra. Then identity* (2) *implies the following identities:*

(3) $$[(x \cup x^*) \cap y^{**}] \cup y^* = [(x \cup x^*) \cup y^*] \cap (y^{**} \cup y^*);$$

(4) $$(x^* \cup x^{**}) \cap (y \cup z) = [(x^* \cup x^{**}) \cap y] \cup [(x^* \cup x^{**}) \cap z].$$

([1]) In a lattice each inclusion can be changed into an identity.

Proof. (3) follows from (2) by an easy calculation. Let (2) hold in L. By Theorem 3 of [3], each sublattice of L generated by the set $\{x, y^*, z\}$ $(x, y, z \in L)$ is distributive. Now it is easy to show that also (4) is true in L.

2.5. *Let V be a variety of modular p-algebras satisfying* (J) *and* (2). *Then, for each non-trivial subdirectly irreducible algebra $L \in V$, there exist a Boolean algebra B and a cardinal number n such that $L \simeq B^n$.*

Proof. Let $0 \neq L \in V$ be subdirectly irreducible. $D(L)$ is a filter and, therefore, a sublattice of L. Assuming that L satisfies (J) and (2), we see that (J) also holds in $D(L)$. By 1.2, $D(L)$ is subdirectly irreducible. Therefore, $D(L) \simeq M_n$ for some cardinal number n (see Theorem 1 in [4]). If $1 \leqslant n \leqslant 2$, then $D(L)$ is distributive. Hence L is also distributive (see 3.2 in [7]). According to 5.2 in [7], $L \simeq B^i$ $(i = 1, 2)$ for some Boolean algebra B. So we can assume $n \geqslant 3$. We know that $y^{**} \cup y^* \in D(L)$. Denote by s the smallest dense element of L. We claim that

(i) $y^* \cup y^{**} = 1$ or $y^* \cup y^{**} = s$.

Suppose $y^* \cup y^{**} = d$ for some $s < d < 1$. Since $n \geqslant 3$, there exist non-comparable elements $s < u$ and $z < 1$ such that $u \neq d \neq z$. Then identity (4) fails in L (take d, u, z). So $y^* \cup y^{**} = 1$ or $y^* \cup y^{**} = s$, as claimed.

Now we prove that

(ii) $y^* \cup y^{**} = 1$ implies $y^* = 0$ or $y^* = 1$.

Suppose on the contrary that $y^* \cup y^{**} = 1$ for $0 \neq y^* \neq 1$. Hence $0 \neq y^{**} \neq 1$. We claim that

(iii) $y^{**} \cup s = d \neq e = y^* \cup s$ $(s < d,\ e < 1)$.

Really, put $a = y^{**}$. Clearly, a is a closed Stone element. Since L is subdirectly irreducible, we infer from 2.1 that $\operatorname{card}(F_a) \geqslant 2$. There exists a $y \in F_a$ such that $y \neq a$. Take an arbitrary $y \in F_a$ with $y \neq a$. By (1), we have

$$y = y^{**} \cap (y \cup y^*) = a \cap (y \cup a^*).$$

Evidently, $y \cup y^* \geqslant s$ because $y \cup y^* \in D(L)$. Therefore,

$$s \leqslant y^* \cup s \leqslant y \cup y^* \leqslant 1.$$

$y \cup y^* = 1$ is impossible, because $y < y^{**} = a$, $y \cap y^* = y^{**} \cap y^* = 0$, and L is modular. Therefore, $y \cup y^* < 1$. Since $y^* = y^{***}$, we have $y^* \in B(L)$. It is easy to see that y^* is a Stone element. By 1.2, we obtain $\operatorname{card}(F_{y^*}) \geqslant 2$. It is known that $t \in F_{y^*}$ implies $t \leqslant y^*$. By (1), we have

$$t = t^{**} \cap (t \cup t^*) = y^* \cap (t \cup t^*).$$

$y^* \cup s = s$ would imply $t \geqslant y^* \cap s = y^*$, because $t \cup t^* \in D(L)$. Hence

$F_{y^*} = \{y^*\}$, a contradiction. Thus $s < y^* \cup s$, and since $D(L) \simeq M_n$, we have established

(iv) $s < e = y^* \cup s = y \cup y^* < 1$ $(y \in F_a, y \neq a)$.

Since $\text{card}(F_{y^*}) \geqslant 2$, we can analogously prove

$$s < d = y^{**} \cup s < 1.$$

However, $1 = y^* \cup y^{**} = d \cup e$ implies $d \neq e$. Thus (iii) holds, as claimed. According to $D(L) \simeq M_n$ $(n \geqslant 3)$, there exists a t $(s < t < 1)$ such that

$$y^{**} \cup s = d \neq t \neq e = y^* \cup s.$$

Evidently, $t = t \cup t^*$ and $y^{**} \cap t \neq y^{**}$ $(y^{**} \cap t = y^{**}$ would imply $y^{**} \cap d \cap t = y^{**} \cap s = y^{**}$, and whence $y^{**} \cup s = s)$. Clearly, we have $y = y^{**} \cap t \in F_a$ for $a = y^{**}$. By (iv), we obtain

(v) $e = [(t \cup t^*) \cap y^{**}] \cup y^* = y \cup y^*$.

On the other hand, $(t \cup t^*) \cup y^* \geqslant t \cup (s \cup y^*) = t \cup e = 1$. Therefore,

(vi) $1 = [(t \cup t^*) \cup y^*] \cap (y^{**} \cup y^*)$.

From (v) and (vi) it follows that identity (3) fails in L, which is a contradiction. Thus the proof of condition (ii) is complete.

Let us consider the sublattice $(s] = \{x \in L; x \leqslant s\}$. It is easy to verify (cf. [1]) that $\langle (s]; \cup, \cap, ^+, 0, s \rangle$ forms a modular p-algebra in which $x^+ = x^* \cap s$ holds. Hence $x^+ = x^* \cap s = 0$ iff $x = s$. Thus $D\big((s]\big) = \{s\}$. By (1), we have $x = x^{++}$ for all $x \in (s]$. Therefore $(s] = B\big((s]\big)$. Hence $(s]$ is a Boolean algebra. It is easy to show that $(s] \simeq B(L)$.

Let $x \in L$. If $x^* = 0$, then $x \geqslant s$. $x^* = 1$ implies $x = 0$. Suppose now $0 < x^* < 1$. By (i) and (iii), we obtain $x \leqslant x^{**} \leqslant s$. Therefore $x \in L$ implies $x \leqslant s$ or $x \geqslant s$. We know that

$$D(L) = \{x \in L; s \leqslant x\} \simeq M_n \quad (n \geqslant 3).$$

If we set $B = (s] \simeq B(L)$, then $L \simeq B^n$, as claimed.

THEOREM 1. *$V(B_m^n)$ is the class of all modular p-algebras satisfying the identities*

(a) (J) *and* (2) *for* $n = \omega = m$;

(b) (J), (J_n) *and* (2) *for* $m = \omega$ *and* $2 \leqslant n \leqslant \omega$;

(c) (J), (L_m) *and* (2) *for* $n = \omega$ *and* $1 \leqslant m < \omega$;

(d) (J), (J_n), (L_m) *and* (2) *for* $2 \leqslant n < \omega$ *and* $1 \leqslant m < \omega$;

(e) (J_2) *and* $x = x^{**}$ *for* $n = 1$.

The proof follows from 2.2 - 2.5 and from the fact that $D(L)$ is a sublattice of L. (e) is a known characterization of Boolean algebras.

Before we formulate the next theorem, we need some notation. For the sake of convenience, (J_ω) will denote identity (J). Analogously, (L_ω) will denote the (trivial) identity $(x \cap x^*)^* = 1$.

l_m or, more precisely, $l_m(x_1, \ldots, x_m)$ will stand for the term

$$(x_1 \cap \ldots \cap x_m)^* \cup (x_1^* \cap \ldots \cap x_m)^* \cup \ldots \cup (x_1 \cap \ldots \cap x_m^*)^*.$$

Substituting the terms $t_0, \ldots, t_n$ into identity (J_n), we get a new identity which will be denoted by $\big(J_n(t_0, \ldots, t_n)\big)$.

THEOREM 2. *$V(B_{m_1}^{n_1}, \ldots, B_{m_r}^{n_r})$ for $r \geqslant 2$ finite and $2 \leqslant n_1 < \ldots < n_r \leqslant \omega$, $\omega \geqslant m_1 > \ldots > m_r \geqslant 1$ is the class of all modular p-algebras satisfying identities* (J), (J_{n_r}), (L_{m_1}), (2) *and*

$$(5i) \qquad \big(J_{n_{i-1}}(l_{m_i} \cup y_0, \ldots, l_{m_i} \cup y_{n_{i-1}})\big) \quad \text{for } i = 2, \ldots, r.$$

Proof. Necessity. According to Theorem 1, we have to check only identity $(5i)$ $(i = 2, \ldots, r)$ for all $B_{m_j}^{n_j}$ $(j = 1, \ldots, r)$. But $l_{m_i} = 1$ in $B_{m_i}^{n_i}, \ldots, B_{m_r}^{n_r}$. Therefore, $(5i)$ holds in every $B_{m_j}^{n_j}$ $(j = i, \ldots, r)$. Since (J_j) implies (J_k) for $j \leqslant k$, $(5i)$ also holds in $B_{m_k}^{n_k}$ $(k = 1, \ldots, i-1)$.

Sufficiency. Let V be a variety of all modular p-algebras satisfying identities (J), (J_{n_r}), (L_{m_1}), (2) and $(5i)$ for $i = 2, \ldots, r$. Let L be a nontrivial subdirectly irreducible algebra in V. By 2.5 and Theorem 1, $L = B_l^k$ for $k \leqslant n_r$ and $l \leqslant m_1$. Let $n_{i-1} < k \leqslant n_i$. We claim that $l \leqslant m_i$. If $l > m_i$, then there exist elements $x_1, \ldots, x_{m_i} \in L$ such that

$$l_{m_i} = l_{m_i}(x_1, \ldots, x_{m_i}) \leqslant t \quad \text{for all } t \in D(L).$$

Choosing distinct atoms $y_0, \ldots, y_{n_{i-1}}$ from $D(L)$, we get $l_{m_i} \cup y_j = y_j$ for all $j = 0, \ldots, n_{i-1}$. But $\big(J_{n_{i-1}}(y_0, \ldots, y_{n_{i-1}})\big)$ fails in L. Therefore $l \leqslant m_i$, as claimed. We have shown that all algebras $B_{m_j}^{n_j}$ $(j = 1, \ldots, r)$ satisfy identities $(5i)$ for $i = 2, \ldots, r$. Hence $V = V(B_{m_1}^{n_1}, \ldots, B_{m_r}^{n_r})$.

3. The lattice of varieties. This section is concerned with the lattice $L(V)$ of all subvarieties of modular p-algebras satisfying identities (J) and (2). Of prime importance for general results is paper [2] by Jónsson.

Since the congruence-relations lattice of every p-algebra is distributive, we have, by 4.1 and 4.2 of [2],

3.1. *$L(V)$ is distributive. Moreover, if $L \in V_1 \cup V_2$, $V_1, V_2 \in L(V)$, L — subdirectly irreducible, then $L \in V_1$ or $L \in V_2$.*

3.2. *In the lattice $L(V)$ the following relations are true:*

(i) $V(B_m^\omega) = \bigvee_{n<\omega} \big(V(B_m^n)\big)$ $(1 \leqslant m \leqslant \omega)$;

(ii) $V(B_\omega^n) = \bigvee_{m<\omega} \big(V(B_m^n)\big)$ $(1 \leqslant n \leqslant \omega)$.

Proof. (i) If an identity does not hold in the algebra B_m^ω, then, by 2.1, it does not hold in a finite subalgebra of B_m^ω, and, consequently, in some subalgebra B_m^n $(n < \omega)$. Therefore

$$V(B_m^\omega) \leqslant \bigvee_{n<\omega} \big(V(B_m^n)\big).$$

Since B_m^n $(n < \omega)$ are subalgebras of B_m^ω, the converse inclusion is trivial. Analogously we prove (ii).

3.3. *For each non-trivial variety $V \in L(V)$, there exists a finite set of p-algebras $\{B_{m_1}^{n_1}, \ldots, B_{m_r}^{n_r}\}$ $(1 \leqslant n_1 < \ldots < n_r \leqslant \omega;\ \omega \geqslant m_1 > \ldots > m_r \geqslant 1)$ such that*

$$V = V(B_{m_1}^{n_1}, \ldots, B_{m_r}^{n_r})$$

holds true.

Proof. Let SI(V) denote the class of all subdirectly irreducible algebras contained in V. If SI$(V) = \{B_1^1\}$, then, clearly, $V = V(B_1^1)$. Suppose now $B_m^2 \in$ SI(V). We know, by 3.2, that there exists, for each $B_m^n \in$ SI(V) $(2 \leqslant n \leqslant \omega)$, the largest number $m(n)$ $(1 \leqslant m(n) \leqslant \omega)$ with $B_{m(n)}^n \in$ SI(V). So we can assign to V the sequence

(i) $$m(2) \geqslant \ldots \geqslant m(n) \geqslant \ldots$$

Put $m_1 = m(2)$. By 3.2, there exists the largest number n_1 $(2 \leqslant n_1 \leqslant \omega)$ such that $m_1 = m(n_1)$. If $m(n_1)$ is the last term of (i), then $V = V(B_{m_1}^{n_1})$. Otherwise, there exists a $B_m^n \in$ SI(V) with $m < m_1$ and $n > n_1$. There exists the largest $m_2 < m_1$ such that $B_{m_2}^n \in$ SI(V) and $n > n_1$. By 3.2, we can find the corresponding largest $n_2 > n_1$ with $B_{m_2}^{n_2} \in$ SI(V). Continuing this process, we get finally the set

(ii) $$\{B_{m_1}^{n_1}, B_{m_2}^{n_2}, \ldots\} \qquad (2 \leqslant n_1 < n_2 < \ldots \leqslant \omega;\ \omega \geqslant m_1 > m_2 > \ldots \geqslant 1)$$

with the following property:

For each $B_m^n \in$ SI(V), there exists a member $B_{m_i}^{n_i}$ of (ii) such that $n \leqslant n_i$ and $m \leqslant m_i$.

Since (i) is non-increasing, set (ii) is finite and our statement is proved.

Remark. According to Theorems 1 and 2 and Proposition 3.3, we can characterize each variety $V \in L(V)$ in terms of identities.

3.4. Corollary ([8]). *For each non-trivial variety V of distributive p-algebras, there exists a p-algebra B_m^n $(1 \leqslant n \leqslant 2;\ 1 \leqslant m \leqslant \omega)$ with $V = V(B_m^n)$.*

Proof. According to Theorem 1, B_1^1 and B_m^2 $(1 \leqslant m \leqslant \omega)$ are the only distributive subdirectly irreducible p-algebras. Therefore, by 3.3, we have $V = V(B_1^1)$ or $V = V(B_m^2)$ $(1 \leqslant m \leqslant \omega)$.

Before formulating the next theorem, we need one more notation. Let L be a lattice. Let $Q(L)$ be a family of subsets of the lattice L having the following properties:

(i) $\emptyset \in Q(L)$;

(ii) for each $a \in L$, $(a] = \{x \in L; x \leqslant a\} \in Q(L)$;

(iii) $Q(L)$ is the smallest family fulfilling (i) and (ii) and closed under formation of set union.

It is easy to show that $Q(L)$, ordered by set inclusion, is a distributive lattice with the smallest element $\emptyset$, and that the lattice operations are the set union and the set intersection.

Let now W_1 denote the chain $1 < 2 < \ldots < n < \ldots < \omega$ and W_2 the chain $2 < 3 < \ldots < n < \ldots < \omega$. Both chains are well ordered of type $\omega+1$. Consider the direct product $W = W_2 \times W_1$. W is a distributive lattice with the smallest element $\langle 2, 1\rangle$ and the largest element $\langle \omega, \omega\rangle$. Set $W_{21} = W \cup \{\langle 1, 1\rangle\}$ and let $x > \langle 1, 1\rangle$ for all $x \in W$. Clearly, W_{21} is a lattice.

3.5. *Let*

$$V_1 = V(B_{m_1}^{n_1}, \ldots, B_{m_r}^{n_r}) \qquad (1 \leqslant n_1 < \ldots < n_r \leqslant \omega;\ \omega \geqslant m_1 > \ldots > m_r \geqslant 1),$$

$$V_2 = V(B_{q_1}^{p_1}, \ldots, B_{q_s}^{p_s}) \qquad (1 \leqslant p_1 < \ldots < p_s \leqslant \omega;\ \omega \geqslant q_1 > \ldots > q_s \geqslant \omega)$$

be varieties of modular p-algebras. Then $V_1 \subseteq V_2$ *in* $L(V)$ *iff*

$$(\langle n_1, m_1\rangle] \cup \ldots \cup (\langle n_r, m_r\rangle] \subseteq (\langle p_1, q_1\rangle] \cup \ldots \cup (\langle p_s, q_s\rangle]$$

in $Q(W_{21})$.

The proof is straightforward.

THEOREM 3. *Let $L(V)$ be the lattice of all subvarieties of modular p-algebras satisfying identities* (J) *and* (2). *Then $L(V)$ is isomorphic to the lattice $Q(W_{21})$.*

Proof. Each class $V \in L(V)$ is uniquely determined by the subclass $SI(V)$. We can establish the isomorphism $\varphi\colon L(V) \to Q(W_{21})$ as follows:

$$\varphi(V) = \begin{cases} \emptyset & \text{if } V \text{ is trivial}, \\ (\langle n_1, m_1\rangle] \cup \ldots \cup (\langle n_r, m_r\rangle] & \text{if } V = V(B_{m_1}^{n_1}, \ldots, B_{m_r}^{n_r}). \end{cases}$$

To show that φ is an isomorphism between $L(V)$ and $Q(W_{21})$ is an easy calculation which can be left to the reader.

3.6. COROLLARY ([8]). *The varieties of distributive p-algebras form the well-ordered chain*

$$V_0 \subset V(B_1^1) \subset V(B_1^2) \subset \ldots \subset V(B_n^2) \subset \ldots \subset V(B_\omega^2)$$

of type $\omega+1$.

The proof follows from 3.4, 3.5 and Theorem 3.

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