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FUNCTIONS WHICH ARE FOURIER TRANSFORMS OF DISTRIBUTIONS

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Let $\mathscr{D}$ and $\boldsymbol{D}$ denote the topological vector spaces of test functions for distributions on R^n and of their Fourier transforms on C^n, respectively. As indicated in [1], p. 135, a locally integrable function F on $R^n \subset C^n$ for which the product $F\Phi$ is integrable over R^n for each $\Phi \in \boldsymbol{D}$ is identified with the transform

$$\Phi \mapsto \int F(x)\Phi(x)\,dx \tag{1}$$

provided this mapping is continuous on $\boldsymbol{D}$. It is our main purpose here to show that continuity, in fact, is always assured in the circumstance of Lebesgue (absolute) integrability.

THEOREM 1. *A locally integrable function F on R^n such that $F\Phi \in L^1$ for each $\Phi \in \boldsymbol{D}$ is the Fourier transform of a distribution.*

Proof. Let F satisfy the assumptions of the theorem. Then let f be the mapping

$$\varphi \mapsto \mathscr{F}^{-1}\big(F\mathscr{F}(\varphi)\big) = f(\varphi) \tag{2}$$

defined on $\mathscr{D}$, where $\mathscr{F}$ and $\mathscr{F}^{-1}$ are the Fourier and inverse Fourier transformations, respectively. For each $\varphi \in \mathscr{D}$ we have $F\mathscr{F}(\varphi) \in L^1$, and so $f(\varphi)$ is the continuous (actually C^∞) function given by

$$f(\varphi)(t) = \int e^{itx}F(x)\Phi(x)\,dx, \tag{3}$$

where $\Phi = \mathscr{F}(\varphi)$. We can show that the mapping f commutes with convolution. Indeed, if $\varphi, \sigma \in \mathscr{D}$, then, by (2),

$$\mathscr{F}\big(f(\varphi*\sigma)\big) = F\mathscr{F}(\varphi*\sigma) = F\big(\mathscr{F}(\varphi)\mathscr{F}(\sigma)\big),$$

while (see [1], Theorem 2)

$$\mathscr{F}\big(f(\varphi)*\sigma\big) = \mathscr{F}\big(f(\varphi)\big)\mathscr{F}(\sigma) = \big(F\mathscr{F}(\varphi)\big)\mathscr{F}(\sigma);$$

hence $f(\varphi*\sigma) = f(\varphi)*\sigma$. It follows from Theorem 1 in [2] that the mapping f on $\mathscr{D}$ is convolution with a fixed distribution, say f, i.e., $f(\varphi) = f*\varphi$ for all $\varphi \in \mathscr{D}$. But then the distribution f and the function F satisfy the Parseval relation (corresponding to (3) with $t = 0$)

$$f * \varphi(0) = \langle f, \check{\varphi} \rangle = \langle F, \Phi \rangle = \int F(x)\Phi(x)\,dx \quad \text{for all } \varphi \in \mathscr{D},$$

where $\check{\varphi}(t) = \varphi(-t)$ and $\Phi = \mathscr{F}(\varphi)$. Thus F is the Fourier transform of the distribution f and mapping (1) is necessarily continuous. This completes the proof of the theorem.

An analogous result holds for tempered distributions with the Schwartz space $\mathscr{S}$ in place of $\boldsymbol{D}$. Since the Fourier transform of a tempered distribution is a tempered distribution, the latter result may be reformulated as follows:

THEOREM 2. *A locally integrable function F on R^n such that $F\Phi \in L^1$ for each $\Phi \in \mathscr{S}$ is a tempered distribution.*

See [3] for some related results.

REFERENCES

[1] L. Ehrenpreis, *Analytic functions and the Fourier transform of distributions, I*, Annals of Mathematics 63 (1956), p. 129-159.

[2] R. A. Struble, *An algebraic view of distributions and operators*, Studia Mathematica 37 (1971), p. 106-109.

[3] Z. Szmydt, *On regular temperate distributions*, ibidem 44 (1972), p. 309-314.

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