COLLOQUIUM MATHEMATICUM

VOL. XXVII 1973 FASC. 2

PREPARACOMPACTNESS AND ℵ-PREPARACOMPACTNESS IN q-SPACES

BY

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The purpose of this paper* is to define two properties, preparacompactness (ppc) and ℵ-preparacompactness (ℵ-ppc), and to compare them with paracompactness and collectionwise normality in various q-spaces. Preparacompactness is a generalization of paracompactness in much the same way that countable compactness is a generalization of compactness. In fact, in the spaces studied, ppc is equivalent to paracompactness only when countable compactness is equivalent to compactness. In a given space, the implications of ppc depend more heavily upon the topological structure of the space than upon the definition. (A ppc regular developable (Moore) space is metrizable, while a ppc regular q-space need not be normal.) ℵ-ppc is a generalization of ppc. Although, in regular q-spaces, ℵ-ppc need not imply ppc, their known implications relative to paracompactness and collectionwise normality are the same. In Hausdorff q-spaces, ℵ-ppc is substantially weaker than ppc. (If a q-space is Lindelöf and ppc, it is paracompact; an ℵ-ppc Lindelöf q-space need not be regular.) Summaries of the relationships between ppc, ℵ-ppc, paracompactness and collectionwise normality are given at the beginning of each numbered section of the paper.

Definitions, terminology and notation. A Hausdorff space S is *preparacompact* (*ℵ-preparacompact*) if and only if, for each open cover G of S, there exists an open refinement H covering S such that if $\{h_\alpha \mid \alpha \in A\}$ is an infinite (uncountable) subcollection of distinct elements of H; if p_α and $q_\alpha \in h_\alpha$, for each $\alpha \in A$, $p_\alpha \neq p_\beta$ and $q_\alpha \neq q_\beta$ for $\alpha \neq \beta$; and if the point set $\{p_\alpha \mid \alpha \in A\}$ has a limit point in S, then the point set $\{q_\alpha \mid \alpha \in A\}$ has a limit point in S.

A topological space S is said to be a *q-space* if and only if, for each point p of S, there exists a sequence $\{N_i\}_{i=1}^{\infty}$ of neighborhoods containing p

* This paper contains a portion of the author's dissertation, University of Houston, August 1968. In the dissertation, ppc is called *strong cover compactness*, and ℵ-ppc — *weak cover compactness*.

such that if $y_i \in N_i$ for each i, and $y_i \neq y_j$ for $i \neq j$, then the point set $\{y_i\}$ has a limit point in S (see [9]).

For the definition of F_σ-screenable, see [6]. For all other definitions, see [10]. All spaces in this paper are assumed to be Hausdorff. If M and N are sets, $M-N = \{p \mid p \in M, p \notin N\}$. If G is a collection of sets, $G^* = \bigcup\{g \mid g \in G\}$. If M is a point set, $\mathrm{Cl}(M)$ is the closure of M. ω denotes the first infinite ordinal, and Ω_1 the first uncountable ordinal.

1. F_σ-screenable q-spaces. In a regular F_σ-screenable q-space, paracompactness, collectionwise normality, ppc and $\aleph$-ppc are equivalent (Theorem 4). It follows that a ppc ($\aleph$-ppc) regular semi-metric space is paracompact, and a ppc ($\aleph$-ppc) Moore space is metrizable (Corollary 4).

LEMMA 1 (Stone [11] and Bing [1]). *If a space is paracompact, it is collectionwise normal.*

LEMMA 2 (McAuley [6]). *If a space is collectionwise normal and F_σ-screenable, it is paracompact.*

LEMMA 3 (Michael [7]). *If a space is collectionwise normal and metacompact, it is paracompact.*

The proofs of Theorems 1 and 2 follow immediately from the definitions.

THEOREM 1. *If a space is paracompact, it is ppc and $\aleph$-ppc.*

THEOREM 2. *If a space is ppc, it is $\aleph$-ppc.*

THEOREM 3. *Let S denote an F_σ-screenable q-space and let G denote an open cover of S. If H is an $\aleph$-ppc refinement of G, then some subcollection of H is a σ-closure preserving open cover of S.*

Proof. Since S is F_σ-screenable, let $X = \{X_i\}_{i=1}^{\infty}$ denote a refinement of H such that, for each i, X_i is a discrete collection of closed sets refining H and $\{X_i^*\}^* = S$. If some countable subcollection of H covers S, it is the desired collection. If not, X is the union of two subcollections X' and X'', where $X_i \in X'$ if and only if some countable subcollection of H covers X_i^*. Clearly, there is a σ-closure preserving subcollection of H covering X'^{**}. We will show the same is true for X'' by constructing, for each $X_i \in X''$, a closure preserving subcollection H_i'' of H which covers X_i^*.

Let X_j denote an element of X'', and, for each $x \in X_j$, choose $h(x)$ to be an element of H containing x. Let $H_j'' = \{h_a \mid h_a = h(x)$ for some $x \in X_j\}$. Assume H_j'' is not closure preserving. Then there is a point $p \in S$ such that $p \in \mathrm{Cl}(\{h_a\}^*) - \{\mathrm{Cl}(h_a)\}^*$. Since S is a q-space and $p \notin \mathrm{Cl}(h_a)$, for any a, there is a countable subcollection $\{h_i\}$ of H_j'' and a sequence of points $\{p_i\}$ in S such that

(1) $p_i \in h_i$ for each i,

(2) $p_i \neq p_k$ and $h_i \neq h_k$ for $i \neq k$, and

(3) the point set $\{p_i\}$ has a limit point p' in S.

Since, for each i, $h_i = h(x_i)$ for some $x_i \in X_j$, let $q_i \in h_i \cap x_i$. Since the elements of $\{h_i\}$ are distinct, the elements of $\{x_i\}$ and $\{q_i\}$ are distinct. It follows that, for each i, p_i and $q_i \in h_i$ and $\{p_i\}$ has a limit point p' in S, while $\{q_i\}$ does not. To obtain a contradiction, we must extend each of these countable sets to uncountable ones.

Since no countable subcollection of H_j'' covers X_j, there is an uncountable subcollection $\{h_a'\}$ of $H_j'' - \{h_i\}$, an uncountable subcollection $\{x_a\}$ of $X_j - \{x_i\}$ and an uncountable point set $\{q_a\}$ in S such that

(1) $h_a' = h(x_a)$ for each a,

(2) $q_a \in h_a' \cap x_a$ for each a, and

(3) $h_a' \neq h_\beta'$, $x_a \neq x_\beta$ and $q_a \neq q_\beta$ for $a \neq \beta$.

It follows that $\hat{H} = \{h_i\} \cup \{h_a'\}$ is an uncountable subcollection of distinct elements of H, $P = \{p_i\} \cup \{q_a\}$ and $Q = \{q_i\} \cup \{q_a\}$ are uncountable point sets, each point belonging to a correspondingly indexed element of H, and P has a limit point p' in S, while Q does not. This is contradictory, hence H_j'' is closure preserving.

COROLLARY 1. *If a regular F_σ-screenable q-space is $\aleph$-ppc, it is paracompact.*

Proof. Since every open cover of S has a σ-closure preserving open refinement, S is paracompact (see [8]).

THEOREM 4. *In a regular F_σ-screenable q-space S, the following statements are equivalent:*

(1) *S is paracompact.*

(2) *S is collectionwise normal.*

(3) *S is ppc.*

(4) *S is $\aleph$-ppc.*

Proof. (1) is equivalent to (2), by Lemmas 1 and 2. (1) $\to$ (3) $\to$ (4) $\to$ (1), by Theorems 1 and 2 and Corollary 1.

COROLLARY 2. *In a regular semi-metric space, paracompactness, collectionwise normality, ppc and $\aleph$-ppc are equivalent, and in a Moore space each is equivalent to metrizability.*

Proof. A semi-metric space is F_σ-screenable (see [6]). A Moore space is a regular semi-metric space (see [3], p. 103-119) and, in a Moore space, paracompactness is equivalent to metrizability (see [1] and [11]).

Note. A paracompact regular semi-metric space need not be metrizable (see [5]).

2. Normal q**-spaces.** In a normal q-space, paracompactness implies ppc, $\aleph$-ppc and collectionwise normality (Lemma 1 and Theorem 1). None of the converse implications hold (Example I).

Both ppc and $\aleph$-ppc imply collectionwise normality (Theorem 5 and Corollary 3), but not conversely (Example II). Example III is given to show that Theorem 5 cannot be extended to normal Hausdorff spaces.

It is not known whether ppc and $\aleph$-ppc are equivalent in normal q-spaces (**P 830**).

THEOREM 5. *If a normal q-space is $\aleph$-ppc, it is collectionwise normal.*

Proof. Let $X = \{X_\alpha \mid \alpha \in A\}$ denote a well-ordered discrete collection of closed point sets in S. (We can assume X is uncountable, since a normal space is collectionwise normal with respect to countable collections.) For each $\alpha \in A$ and each point $p \in X_\alpha$, let O_p denote an open set containing p such that $\mathrm{Cl}(O_p) \cap (X - X_\alpha)^* = \emptyset$. Then $\{O_p \mid p \in X_\alpha, \alpha \in A\} \cup (S - X^*)$ is an open cover of S and has an $\aleph$-ppc refinement H. For each $\alpha \in A$, let $H_\alpha = \{h \mid h \in H, h \cap X_\alpha \neq \emptyset\}$.

We now prove that for each $X_\alpha \in X$; we can construct an open set $D_\alpha \supset X_\alpha$ as follows.

Let $B_\alpha = [\{D_\beta \mid \beta < \alpha\} \cup (X - \{X_\beta \mid \beta \leqslant \alpha\})]^*$. Let D'_α and D''_α denote open sets containing $\mathrm{Cl}(B_\alpha)$ and X_α, respectively, such that $\mathrm{Cl}(D'_\alpha) \cap \mathrm{Cl}(D''_\alpha) = \emptyset$. Let $D_\alpha = D'_\alpha \cap H^*_\alpha$. It is easily shown that if D_α can be constructed for each $\alpha \in A$, then $D = \{D_\alpha \mid \alpha \in A\}$ is the desired collection of open sets covering X.

Assume that, for some $X_\alpha \in X$, D_α cannot be constructed. Let X_γ denote the first such element. Then, since S is normal, X_γ and $\mathrm{Cl}(B_\gamma)$ are not mutually exclusive. Since, for $\beta < \gamma$, $X_\gamma \cap \mathrm{Cl}(D_\beta) = \emptyset$, there is a point $p_\gamma \in X_\gamma$ such that $p_\gamma \in \mathrm{Cl}(\{D_\beta \mid \beta < \gamma\}^*) - \{\mathrm{Cl}(D_\beta) \mid \beta < \gamma\}^*$. Since S is a q-space and $p_\gamma \notin \mathrm{Cl}(D_\beta)$, for any $\beta < \gamma$, there is a countable subcollection $\{D_i\}$ of $\{D_\beta \mid \beta < \gamma\}$ and a sequence of points $\{p_i\}$ in S such that

(1) $p_i \in D_i$ for each i,

(2) $p_i \neq p_j$ and $D_i \neq D_j$ for $i \neq j$, and

(3) the point set $\{p_i\}$ has a limit point p'_γ in S.

For each i, $p_i \in h_i$, for some $h_i \in H_i$. Let $\{h_i\}$ denote such a subcollection of H and, for each i, let $q_i \in h_i \cap X_i$. Since (by construction) no element of H belongs to both H_i and H_j, $i \neq j$, $\{h_i\}$ is an infinite collection of distinct elements of H; p_i and $q_i \in h_i$ for each i, and $\{p_i\}$ has a limit point p'_γ in S, while $\{q_i\}$ does not. We extend these infinite collections to uncountable collections to obtain a contradiction. For each $X_\alpha \in X - \{X_i\}$, let $h_\alpha \in H_\alpha$ and let $q_\alpha \in h_\alpha \cap X_\alpha$. Then $\hat{H} = \{h_i\} \cup \{h_\alpha\}$ is an uncountable subcollection of distinct elements of H, but the point sets $P = \{p_i\} \cup \{q_\alpha\}$ and $Q = \{q_i\} \cup \{q_\alpha\}$ contradict the fact that H is an $\aleph$-ppc refinement. Hence, D_α can be constructed for each $X_\alpha \in X$, and S is collectionwise normal.

COROLLARY 3. *If a normal q-space is ppc, it is collectionwise normal.*

Example I. A countably compact, first countable, normal q-space which is ppc, $\aleph$-ppc and collectionwise normal, but not paracompact.

Let $S = \{a \mid a < \Omega_1\}$, with the usual order topology. It is well known that S is a countably compact (hence, ppc and $\aleph$-ppc), first countable, collectionwise normal q-space which is not paracompact (see [1]).

Example II. A first countable collectionwise normal q-space which is neither ppc nor $\aleph$-ppc.

Let S_1 denote the space in Example I and let $S_2 = \{i \mid i \leqslant \omega\}$, with the order topology (discrete except at ω). Let $S = S_1 \times S_2$, with the product topology at each point of the form (a, ω), $a < \Omega_1$, and with discrete topology elsewhere.

S is clearly first countable. Let $S' = \{(a, \omega) \mid a < \Omega_1\}$. Since the subspace S' is simply S_1, and since every closed subset of $S - S'$ is both open and closed, it is easily shown that S is collectionwise normal. We now prove that S is neither ppc nor $\aleph$-ppc. For each point of the form (a, ω), $a < \Omega_1$, let $O_a = \{(\gamma, j) \mid 1 \leqslant \gamma \leqslant a, 1 \leqslant j \leqslant \omega\}$. Let $O = \{O_a \mid a < \Omega_1\}$ and let H denote any open refinement of O. Clearly, no countable subcollection of H covers S', hence no countable subcollection of H covers S. Therefore, there exists a well-ordered uncountable subcollection H' of H such that if $h_a \in H'$, there is a point $(\delta_a, \omega) \in h_a$ such that $(\delta_a, \omega) \notin h_\beta$ for any $\beta < a$. For each point (δ_a, ω) thus defined, let i_a denote the smallest integer such that if $j > i_a$, $(\delta_a, j) \in h_a$. Since H' is uncountable, there is an integer k and an uncountable subcollection H'' of H' such that if $h_a \in H''$, (δ_a, ω) and $(\delta_a, k+1) \in h_a$. Consider the sets $\{(\delta_a, \omega) \mid h_a \in H''\}$ and $\{(\delta_a, k+1) \mid h_a \in H''\}$. The former is a subset of S', and hence has a limit point in S; the latter is a subset of $S - S'$ and has no limit point. The desired conclusion follows.

The following space is a slight modification of an example by Bing [1], Example G, and Michael [7], Example 2:

Example III. A normal, metacompact space which is ppc and $\aleph$-ppc, but not collectionwise normal.

Let P denote a point set of cardinality $\aleph_1$ and let T denote the collection of all subsets of P. For each $p \in P$, let f_p denote the following function defined on T: $f_p(t) = 1$ if $p \in t$, and $f_p(t) = 0$ if $p \notin t$. Let $F_p = \{f_p \mid p \in P\}$. Let F' denote the collection of all functions f defined on T such that $f(t) = 0$ or 1 for only finitely many $t \in T$, and $f(t) = 2$ otherwise. Let $F = F_P \cup F'$, and let S denote the space whose points are the elements of F. Let the topology of S be defined as follows:

(1) if $f \in F'$, let f be a degenerate open set;

(2) if $f_p \in F_p$, and r is a finite subcollection of T, say $r = \{t_1, t_2, \ldots, t_n\}$, let $r_p = \{f \mid f \in S$ and $f(t_i) = f_p(t_i)$, $1 \leqslant i \leqslant n\}$ denote an r-neighborhood of f_p.

The proof that S is normal but not collectionwise normal is the same as Bing's [1]. The proof that S is metacompact is the same as Michael's [7].

We now prove that S is ppc. Since any open cover of S can be refined by a cover having at most $\aleph_1$ non-degenerate open sets, it is sufficient to prove that no subset of S having cardinality less than or equal to $\aleph_1$ has a limit point. Assume there is a subset $\{f_a \mid a \in A\}$ of S having cardinality less than or equal to $\aleph_1$ such that $\{f_a\}$ has a limit point, say f_q. (We can assume that $\{f_a\} \subset F'$, since F_P has no limit point in S.) For each $a \in A$, let $T_a = \{t_i \mid 1 \leqslant i \leqslant n_a\}$ denote the elements of T such that $f_a(t_i) = 0$ or 1. Since each T_a is a finite collection, the cardinality of $\bigcup T_a \leqslant \aleph_1$, but the cardinality of T is $2^{\aleph_1}$. Hence there is an element t_0 in $T - \bigcup T_a$ such that $f_a(t_0) = 2$ for each $a \in A$. It follows that the r-neighborhood $\{t_0\}_q$ of f_q contains no point of $\{f_a\}$, contradicting our assumption. Thus S is ppc and $\aleph$-ppc.

3. Regular and metacompact regular q-spaces. In a regular q-space, paracompactness implies ppc, $\aleph$-ppc and collectionwise normality (Lemma 1 and Theorem 1). None of the converse implications hold (Example I). Moreover, ppc implies $\aleph$-ppc, but not conversely (Theorem 2 and Example IV). Neither implies collectionwise normality (nor even normality) (Example V). Collectionwise normality does not imply any of the other properties (Examples I and II).

In a metacompact regular q-space, paracompactness, ppc, $\aleph$-ppc and collectionwise normality are equivalent (Theorem 6). Example III shows that Theorem 6 cannot be extended to metacompact regular Hausdorff spaces.

Example IV. A regular locally countably compact q-space which is $\aleph$-ppc, but not ppc.

Let $S_1 = \{a \mid a \leqslant \Omega_1\}$, with the usual order topology. Let S_2 be defined as in Example II and let $S = (S_1 \times S_2) - (\Omega_1, \omega)$, with the product topology. Clearly, S is a regular locally countably compact q-space and, since every uncountable subset of S has a limit point, S is $\aleph$-ppc. We now prove that S is not ppc. Let $C = \{(\Omega_1, i) \mid i < \omega\}$ and, for each i, let O_i denote an open set containing (Ω_1, i), but no other point of C. $\{O_i\} \cup \cup (S - C)$ is an open cover of S. Let H denote any open refinement of this cover and, for each i, let h_i denote an element of H containing (Ω_1, i). For each i, there is an ordinal a_i such that if $\beta > a_i$, $(\beta, i) \in h_i$. Since $\{a_i\}$ is countable, there is a $\delta < \Omega_1$ such that $\delta > a_i$ for every i. It follows that, for each i, (δ, i) and $(\Omega_1, i) \in h_i$, and $\{(\delta, i)\}$ has a limit point (δ, ω) in S, while $\{(\Omega_1, i)\}$ does not.

Example V. A regular countably compact q-space which is ppc and $\aleph$-ppc, but not normal.

Let S_1 be defined as in Example IV. Let $S_2 = S_1 - \Omega_1$, and let $S = S_1 \times S_2$, with the product topology. S is countably compact (hence, ppc and $\aleph$-ppc), but not normal (see [4], p. 131).

THEOREM 6. *In a metacompact regular q-space S, the following statements are equivalent:*

(1) *S is paracompact.*

(2) *S is collectionwise normal.*

(3) *S is ppc.*

(4) *S is $\aleph$-ppc.*

Proof. (1) is equivalent to (2), by Lemmas 1 and 3. (1)$\to$(3)$\to$$\to$(4), by Theorems 1 and 2. To complete the proof, we show that (4) implies (1).

If S is a Lindelöf space, S is paracompact. If not, there is an open cover G of S such that no countable subcollection of G covers S. It is sufficient to show that such a cover has a locally finite refinement. Let H denote an $\aleph$-ppc refinement of G, and let K denote a point-finite open refinement of H. Let H be well ordered and let D_a denote the collection to which an element k of K belongs if and only if h_a is the first element of H such that $k \subset h_a$. Let O denote the collection to which an element O_a belongs if and only if $O_a = D_a^*$, $D_a^* \neq \emptyset$. O is a point-finite collection of open sets refining G and covering S. Moreover, if O_a and $O_\beta \in O$, $a \neq \beta$, there exist elements h_a and $h_\beta \in H$, $h_a \neq h_\beta$, such that $O_a \subset h_a$ and $O_\beta \subset h_\beta$. Hence O is also an $\aleph$-ppc refinement of G. Let O' denote a minimal subcollection of O covering S (see [2], p. 160). We now prove that O' is locally finite.

Assume O' is not locally finite at some point $p \in S$. Since S is a q-space, there is a countable subcollection $\{O_i\}$ of O' and a sequence of points $\{p_i\}$ in S such that

(1) $p_i \in O_i$ for each i,

(2) $p_i \neq p_j$ and $O_i \neq O_j$ for $i \neq j$, and

(3) the point set $\{p_i\}$ has a limit point p' in S.

Let $\{O_\beta \mid \beta \in B\}$ denote an uncountable subcollection of $O' - \{O_i\}$. Since O' is minimal, for each integer i and each $\beta \in B$, let $q_i \in O_i - (O' - \{O_i\})^*$ and $q_\beta \in O_\beta - (O' - \{O_\beta\})^*$. Clearly, $\{q_i\} \cup \{q_\beta\}$ has no limit point in S. Hence, the subcollections $\{O_i\} \cup \{O_\beta\}$ and the point sets $P = \{p_i\} \cup \{q_\beta\}$ and $Q = \{q_i\} \cup \{q_\beta\}$ contradict the fact that O' is an $\aleph$-ppc refinement. Thus O' is locally finite and S is paracompact.

4. q-spaces and metacompact q-spaces. In a q-space, paracompactness implies each of the other properties (Lemma 1 and Theorem 1), and ppc implies $\aleph$-ppc (Theorem 2). No other implications hold (Examples I, II, IV and V).

In a metacompact q-space, paracompactness, ppc and collectionwise normality are equivalent (Theorem 7). Each implies $\aleph$-ppc, but not conversely (Theorem 1 and Example VI).

THEOREM 7. *In a metacompact q-space S, the following statements are equivalent:*

(1) *S is paracompact.*

(2) *S is collectionwise normal.*

(3) *S is ppc.*

Proof. (1) is equivalent to (2), by Lemmas 1 and 3. (1) $\rightarrow$ (3), by Lemma 1. We now prove that (3) $\rightarrow$ (1).

Let G denote an open cover of S. Using the same construction employed in Theorem 6, we obtain an open refinement O' of G which is point-finite, ppc, and minimal with respect to covering S. Continuing the construction in Theorem 6, we obtain countable point sets $\{p_i\}$ and $\{q_i\}$ such that $\{p_i\}$ has a limit point p' in S, while $\{q_i\}$ does not.

Example VI. A metacompact, first countable, locally countably compact, Lindelöf q-space which is $\aleph$-ppc, but not regular.

Let $M = \{1/i\}_{i=1}^{\infty}$ and let $S = [0,1] \cup ([0,1] \times M)$, with the relative plane topology except at $(0,0)$. Let O be an open set in S containing $(0,0)$ if and only if there is an open set O' in the plane such that $O = (O' \cap [0,1]) \cup (O' \cap [(0,1] \times M])$. It is easily shown that S is a metacompact, first countable, locally countably compact, Lindelöf q-space. Since every uncountable subset of S has a limit point, S is $\aleph$-ppc. S is not regular with respect to $(0,0)$ and $N = \{(0,1/i)\}_{i=1}^{\infty}$.

5. Lindelöf q-spaces. Since a regular Lindelöf space is paracompact, collectionwise normality is equivalent to paracompactness in Lindelöf q-spaces. Preparacompactness is also equivalent to paracompactness (Theorem 8), but an $\aleph$-ppc Lindelöf q-space need not be regular (Example VI).

THEOREM 8. *If a Lindelöf q-space is ppc, it is paracompact.*

Proof. We will prove that S is regular, and hence paracompact. Let $p \in S$ and let K denote a closed set in S not containing p. For each point $k \in K$, let O_k denote an open set containing k such that $p \notin \mathrm{Cl}(O_k)$. Since $\{O_k \mid k \in K\} \cup (S-K)$ is an open cover of S, it has a ppc refinement H. If some finite subcollection H' of H covers K, $D = H'^*$ is the desired open set containing K. If not, let $\{h_i\}$ denote a countably infinite subcollection of H covering K such that, for each i, h_i contains a point q_i, $q_i \notin h_j$, for $j < i$. Since $\{h_i\}$ covers K, K is closed, and no element of $\{h_i\}$ contains infinitely many points of $\{q_i\}$, $\{q_i\}$ has no limit point in S.

Assume $p \in \mathrm{Cl}(\{h_i\}^*)$. Then $p \in \mathrm{Cl}(\{h_i\}^*) - \{\mathrm{Cl}(h_i)\}^*$. Since S is a q-space, there is a countably infinite subcollection $\{h'_i\}$ of $\{h_i\}$ and a sequence of points $\{p_i\}$ in S such that

(1) $p_i \in h_i$ for each i,

(2) $p_i \neq p_j$ and $h'_i \neq h'_j$ for $i \neq j$, and

(3) the point set $\{p_i\}$ has a limit point p' in S.

For each i, let q'_i denote the point of $\{q_i\}$ belonging to h'_i. Then $\{q'_i\}$ and $\{p_i\}$ contradict the fact that H is a ppc refinement.

QUESTION. Is every regular, first countable, ppc ($\aleph$-ppc) space normal? (If so, it is also collectionwise normal.) (**P 831**)

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Reçu par la Rédaction le 8. 10. 1971