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A UNIFORMLY BOUNDED REPRESENTATION ASSOCIATED TO A FREE SET IN A DISCRETE GROUP

BY

GERO FENDLER (SAARBRÜCKEN)

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Since then analytic series of representations of free groups have been constructed by Pytlik and Szwarc [Py–Sz] and independently by Figà-Talamanca and Picardello [FT–Pi] and Mantero and Zappa [Ma–Za], which yield information also on nonunitarizable representations.

The methods of this note are discussed further by Bożejko in [Bo1] and [Bo2] and are closely related to those used in the author's joint paper with Cowling [Co–Fe].

This note has been part of the author's dissertation written under the supervision of Professor M. Leinert at the University of Heidelberg in 1987 and we take this chance to thank him and Professor M. Bożejko for various discussions on the topic of this note as well as on many others.

1. Introduction. Let G be a locally compact group, $A(G) = A_2(G)$ its Fourier algebra, $B(G)$ its Fourier–Stieltjes algebra, as defined by Eymard [Eym], and $B_2(G)$ its generalized Fourier–Stieltjes algebra in the sense of Herz [Her].

We recall that if (π, H_π) is a uniformly bounded strongly continuous representation of G on a Hilbert space, then any matrix coefficient φ of π, i.e., $\varphi = (\pi(\cdot)\xi, \eta)$ for some $\xi, \eta \in H_\pi$, is an element of $B_2(G)$. If G is amenable, then furthermore $B_2(G)$ coincides with $B(G)$.

On the other hand, the free group on two generators F_2 is not amenable and the characteristic function of an infinite set $E \subset F_2$ which satisfies Leinert's condition:

(*) $\forall n \in N$, $\forall x_1, \ldots, x_{2n} \in E$, if $x_i \neq x_{i+1}$, $i = 1, \ldots, 2n-1$, then

$$x_1^{-1} x_2 x_3^{-1} x_4 \ldots x_{2n-1}^{-1} x_{2n} \neq e,$$

is an example of a function which belongs to $B_2(F_2)$ but not to $B(F_2)$ (cf. [Le2], Korollar 12). This condition is weaker than requiring E to be a free set, by which we mean the following:

$\forall n \in N$, $\forall x_1, \ldots, x_n \in E$, $\forall \varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$, if $x_i \neq x_{i+1}$ or $\varepsilon_i = \varepsilon_{i+1}$, $i = 1, \ldots, n-1$, then

$$x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n} \neq e.$$

We refer the reader to the paper of Akemann and Ostrand [Ak–Os] for a discussion of the relation between free sets and sets satisfying Leinert's condition (*).

As indicated in the title we shall consider only a fixed discrete group G. We show, using a theorem of M. Leinert on the decomposition of certain matrices, the following theorem:

1.1. THEOREM. *Let $E \subset G$ be a set satisfying Leinert's condition (*). Then its characteristic function is a matrix coefficient of a uniformly bounded Hilbert space representation of G.*

From the above discussion it is clear that this representation is not unitarizable if the set E is infinite.

Now we introduce some notation.

Let $K(V)$ denote the space of all finitely supported complex-valued functions on a set V.

If $F \subset V$, let

I_F: $l^1(F) \to l^1(V)$ be the natural inclusion,

J_F: $l^\infty(V) \to l^\infty(F)$ be its transposed operator, and

P_F: $l^1(V) \to l^1(F)$ be the projection defined by

$$P_F: \sum_{y \in V} f(y) e_y \to \sum_{y \in F} f(y) e_y.$$

Finally, for $1 \leqslant p \leqslant \infty$ we denote by ϱ the right regular representation of G on $l^p(G)$:

$$\varrho(s) f(x) = f(xs), \qquad s, x \in G, \ f \in l^p(G).$$

2. Decompositions of certain matrices. Let $E \subset G$ be a set satisfying Leinert's condition (*) and let χ denote its characteristic function. For a finite set $F \subset G$ the operator

$$\lambda(\chi): f \to \chi * f, \qquad f \in l^p(G),$$

induces an operator

$$H_F: l^1(F) \xrightarrow{I_F} l^1(G) \xrightarrow{\lambda(\chi)} l^\infty(G) \xrightarrow{J_F} l^\infty(F),$$

where the matrix of H_F has the entries

$$h_F(y, x) = \langle H_F e_x, e_y \rangle = \chi(yx^{-1}), \qquad x, y \in F.$$

The proofs of Korollar 12 and Satz 9 of [Le2] show that h_F has an additive

decomposition $h_F = q_F + r_F$, where q_F is a row matrix and r_F is a column matrix, i.e., for any $z \in F$ there exists at most one $z' \in F$ such that

$$q_F(z', z) = \langle Q_F e_z, e_{z'} \rangle \neq 0, \quad r_F(z, z') = \langle R_F e_{z'}, e_z \rangle \neq 0.$$

Further we may suppose that the nonzero entries of q_F and r_F are disjoint and equal to one.

Remarks. (i) Q_F has norm at most one when considered as an operator from $l^1(F)$ to $l^2(F)$, and $\|R_F\| \leqslant 1$ as an operator from $l^2(F)$ to $l^\infty(F)$.

(ii) Let $H(F)$ be the l^2-direct sum $l^2(F) \oplus_2 l^2(F)$ and let

$$A_F: l^1(F) \to H(F), \quad C_F: H(F) \to l^\infty(F)$$

be defined by

$$A_F(f) = (Q_F f, f), \ f \in l^1(F), \quad C_F(g, f) = g + R_F(f), \ (g, f) \in H(F).$$

Then, clearly, $H_F = C_F \circ A_F$ and $\|A_F\| \leqslant \sqrt{2}$, $\|C_F\| \leqslant \sqrt{2}$.

(iii) We note that $\|A_F f\|_{H(F)} \geqslant \|f\|_2$, $f \in l^2(F)$.

3. Further implications of the decompositions. Throughout this section let $F \subset G$ be finite, $s \in G$, and fix a finite set $F^\circ \subset G$ with $F^\circ \supset F \cup Fs$. Let d be an abstract symbol. Since in any row of r_F, respectively of r_{F°, there is at most one nonzero entry, there exist functions

$$': F \to F \cup \{d\} \quad \text{and} \quad '': F \to F \cup \{d\}$$

such that, for $z \in F$,

$$r_{F^\circ}(ys, zs) = 1 \quad \text{if and only if} \quad z = y' \in F,$$

$$r_F(y, z) = 1 \quad \text{if and only if} \quad z = y'' \in F.$$

Since the operator $\lambda(\chi)$ commutes with translations from the right the following holds true:

3.1. LEMMA. (i) *For $f \in l^1(F)$ define $f(d) = 0$. Then for $y \in F$*

$$|\langle Q_F f, e_y \rangle| \leqslant |\langle Q_{F^\circ} \varrho(s^{-1}) f, e_{ys} \rangle| + |f(y') - f(y'')|.$$

(ii) *For any $z \in F$ each of the sets*

$$n_z = \{x \in F \mid x' = z, x'' \neq z\}, \quad m_z = \{x \in F \mid x'' = z, x' \neq z\}$$

contains at most one element.

Proof. (i) We have only to note that

$$\begin{aligned} \langle Q_F f, e_y \rangle + f(y'') &= \langle Q_F f, e_y \rangle + \langle R_F f, e_y \rangle \\ &= \langle H_F f, e_y \rangle = \langle \varrho(s) \lambda(\chi) \varrho(s^{-1}) f, e_y \rangle \\ &= \langle H_{F^\circ} \varrho(s^{-1}) f, \varrho(s^{-1}) e_y \rangle \end{aligned}$$

$$= \langle Q_F \circ \varrho(s^{-1})f, e_{ys}\rangle + \langle R_F \circ \varrho(s^{-1})f, e_{ys}\rangle$$
$$= \langle Q_F \circ \varrho(s^{-1})f, e_{ys}\rangle + f(y').$$

(ii) Suppose $x, y \in F$, $x \neq y$, $x' = z$, $y' = z$. Then

$$r_{F^\circ}(xs, zs) = r_{F^\circ}(ys, zs) = 1,$$

so

$$h_{F^\circ}(xs, zs) = h_F(x, z) = 1, \qquad h_{F^\circ}(ys, zs) = h_F(y, z) = 1.$$

Since q_F is a row matrix, we must have $q_F(x, z) = 0$ or $q_F(y, z) = 0$. Thus $r_F(x, z) = 1$ or $r_F(y, z) = 1$, which implies $x'' = z$ or $y'' = z$. So either $x \notin n_z$ or $y \notin n_z$.

The other statement is proved similarly.

3.2. COROLLARY. *Let $f \in K(G)$, $s \in G$, and let $F \supset \operatorname{supp}(f)$. Then for any finite set $F^\circ \supset F \cup Fs$*

$$\|Q_F f\|_2^2 + \|f\|_2^2 \leqslant 9(\|Q_{F^\circ} \varrho(s^{-1})f\|_2^2 + \|\varrho(s^{-1})f\|_2^2).$$

Proof. From (ii) of Lemma 3.1 we obtain

$$\sum_{\substack{x \in F \\ x' \neq x''}} |f(x')|^2 \leqslant \sum_{z \in F} \# n_z |f(z)|^2 \leqslant \|f\|_2^2,$$

$$\sum_{\substack{x \in F \\ x' \neq x''}} |f(x'')|^2 \leqslant \sum_{z \in F} \# m_z |f(z)|^2 \leqslant \|f\|_2^2,$$

where # denotes the cardinality of a set.

On the other hand, an application of (i) then yields

$$\|Q_F f\|_2^2 \leqslant \sum_{x \in F} (|\langle Q_{F^\circ} \varrho(s^{-1})f, e_{xs}\rangle| + |f(x') - f(x'')|)^2$$

$$\leqslant \sum_{x \in F} (2|\langle Q_{F^\circ} \varrho(s^{-1})f, e_{xs}\rangle|^2 + 2|f(x') - f(x'')|^2)$$

$$= 2 \sum_{x \in F} |\langle Q_{F^\circ} \varrho(s^{-1})f, e_{xs}\rangle|^2 + 2 \sum_{x \in F'} |f(x') - f(x'')|^2$$

$$\leqslant 2 \sum_{x \in F^\circ} |\langle Q_{F^\circ} \varrho(s^{-1})f, e_x\rangle|^2 + 2 \sum_{x \in F'} (2|f(x')|^2 + 2|f(x'')|^2)$$

$$\leqslant 8\Big[\sum_{x \in F^\circ} |\langle Q_{F^\circ} \varrho(s^{-1})f, e_x\rangle|^2 + \|f\|_2^2\Big],$$

where $F' = \{x \in F \mid x' \neq x''\}$.

4. Construction of the representation. Let X be the system of finite subsets of G ordered by inclusion, let Ω denote an ultrafilter on X containing the filter

$$\Phi = \{V_{F^\circ} \mid V_{F^\circ} = \{F \in X \mid F \supset F^\circ\}, F^\circ \in X\}$$

generated by the ordering of X.

With respect to Ω we define a sesquilinear form $(\cdot, \cdot)_\Omega$ on $K(G)$ as

$$(f, g)_\Omega = \lim_\Omega (A_F P_F f, A_F P_F g)_{H(F)}.$$

Since $\Phi \subset \Omega$ and since $\|f\|_2 \leqslant \|A_F P_F f\|_{H(F)}$ for all $f \in K(G)$ whenever F is sufficiently large, i.e., F contains the support of $f \in K(G)$, the form $(\cdot, \cdot)_\Omega$ is positive definite on $K(G)$. So $K(G)$ with the norm $\|\cdot\|_\Omega$ corresponding to $(\cdot, \cdot)_\Omega$ is a pre-Hilbert space and

$$\|f\|_\Omega = \lim_\Omega \|A_F P_F f\|, \quad f \in K(G).$$

We are going to show that right translations act uniformly boundedly with respect to this norm.

4.1. LEMMA. *Let $f \in K(G)$, $s \in G$. If a and b are accumulation points of the nets*

$$\{\|A_F P_F \varrho(s) f\|_{H(F)}\}_{F \in X} \quad \text{and} \quad \{\|A_F P_F f\|_{H(F)}\}_{F \in X},$$

respectively, then $a \leqslant 3b$.

Proof. Let $\varepsilon > 0$. Then there exists $F^\circ \supset \operatorname{supp}(\varrho(s)f)$ such that

$$|a - \|A_{F^\circ} P_{F^\circ} \varrho(s) f\|_{H(F^\circ)}| \leqslant \varepsilon/2.$$

Since b is an accumulation point of the net $\{\|A_F P_F f\|_{H(F)}\}_{F \in X}$, there exists an $F' \supset F^\circ \cup F^\circ s$ such that

$$|b - \|A_{F'} P_{F'} f\|_{H(F')}| \leqslant \varepsilon/6.$$

By Corollary 3.2 we have

$$\|A_{F^\circ} P_{F^\circ} \varrho(s) f\|_{H(F^\circ)} \leqslant 3 \|A_{F'} \varrho(s^{-1}) P_{F^\circ} \varrho(s) f\|_{H(F')} \leqslant 3 \|A_{F'} P_{F'} f\|_{H(F')},$$

and so

$$a \leqslant \|A_{F^\circ} P_{F^\circ} \varrho(s) f\|_{H(F^\circ)} + \varepsilon/2 \leqslant 3 \|A_{F'} P_{F'} f\|_{H(F')} + \varepsilon/2 \leqslant 3b + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we obtain $a \leqslant 3b$.

Clearly, for any bounded net $\{a_F\}_{F \in X}$ of real numbers, $\lim_\Omega a_F$ is an accumulation point of the net. Hence the lemma gives

$$\|\varrho(s) f\|_\Omega \leqslant 3 \|f\|_\Omega, \quad f \in K(G), \ s \in G.$$

Therefore ϱ extends to a uniformly bounded representation ϱ' of G on the completion H of $K(G)$ with respect to $\|\cdot\|_\Omega$. It remains to show that the characteristic function χ of E appears as a matrix coefficient of ϱ'. To this end we define a linear functional η^1 on $K(G)$ by

$$\eta^1(f) = \lim_{\Omega} \langle C_F A_F P_F f, P_F e_e \rangle.$$

Then $|\eta^1(f)| \leqslant \sqrt{2}\|f\|_\Omega$, and so there exists some $\eta \in H$ with

$$\eta^1(f) = (f, \eta)_\Omega, \quad f \in K(G), \ \|\eta\| \leqslant \sqrt{2}.$$

Furthermore, if $s \in G$, $s^{-1}, e \in F$, then

$$\chi(s) = \langle \lambda(\chi) e_{s^{-1}}, e_e \rangle = \langle C_F A_F P_F \varrho(s) e_e, P_F e_e \rangle.$$

Since $\Phi \subset \Omega$, this yields $\chi(s) = (\varrho'(s) e_e, \eta)_\Omega$.

4.2. COROLLARY. *Let $E \subset G$ be a free set and $b \in l^\infty(E)$. Then there exist a uniformly bounded Hilbert space representation (π, H) and $\xi, \eta \in H$ such that*

$$(\pi(x)\xi, \eta) = \begin{cases} b(x), & x \in E, \\ 0, & x \notin E. \end{cases}$$

Proof. For $x \in E$ we define a unitary operator on $l^2(\mathbf{Z})$ by displaying its matrix

$$\begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & \ddots & & & & & \\ \ddots & \ddots & & & & & \\ & & 0 & 1 & & & \\ & & 0 & (A) & & & \\ & & & 0 & 1 & & \\ & & & & 0 & \ddots & \\ & & & & & \ddots & \\ & & & & & & \ddots \end{pmatrix},$$

where

$$(A) = \begin{pmatrix} a_{-1,1} & a_{-1,1} \\ a_{0,0} & a_{0,1} \end{pmatrix} = \begin{pmatrix} \sqrt{1-|b(x)|^2} & \overline{-b(x)} \\ b(x) & \sqrt{1-|b(x)|^2} \end{pmatrix}$$

(cf. [Fo–Na], Chapter I, Section 5).

Since E is a free set, we may define a unitary representation of the subgroup $\langle E \rangle$ generated by E by sending $x \in E$ to the above-defined operator. The representation of G induced by this representation of $\langle E \rangle$ has a matrix coefficient which is equal to b on E.

From our theorem we infer that the characteristic function of E is a matrix coefficient of a uniformly bounded representation of G. The function which is equal to b on E and zero elsewhere is then a matrix coefficient of the tensor product of these two representations of G.

REFERENCES

[Ak–Os] Ch. A. Akemann and Ph. A. Ostrand, *Computing norms in group C*-algebras*, Amer. J. Math. 98 (1976), pp. 1015–1047.

[Bo1] M. Bożejko, *Littlewood functions, Hankel multipliers and power bounded operators on a Hilbert space*, Colloq. Math. 51 (1987), pp. 35–42.

[Bo2] — *Uniformly bounded representations of the free group*, J. Reine Angew. Math. 377 (1987), pp. 170–186.

[Co–Fe] M. Cowling and G. Fendler, *On representations in Banach spaces*, Math. Ann. 266 (1984), pp. 307–315.

[Eym] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France 92 (1964), pp. 181–236.

[FT-Pi] A. Figà-Talamanca and M. Picardello, *Spherical functions and harmonic analysis on free groups*, J. Funct. Anal. 47 (1982), pp. 281–304.

[Fo–Na] C. Foiaş and B. Szökefalvi-Nagy, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, Amsterdam 1970.

[Her] C. Herz, *Une généralisation de la notion de transformée de Fourier–Stieltjes*, Ann. Inst. Fourier (Grenoble) 24 (1974), pp. 145–157.

[Le1] M. Leinert, *Faltungsoperatoren auf gewissen diskreten Gruppen*, Studia Math. 52 (1974), pp. 149–158.

[Le2] — *Abschätzung von Normen gewisser Matrizen und eine Anwendung*, Math. Ann. 240 (1979), pp. 13–19.

[Ma–Za1] A. M. Mantero and A. Zappa, *The Poisson transform and representations of free groups*, J. Funct. Anal. 51 (1983), pp. 372–400.

[Ma–Za2] — — *Uniformly bounded representations and L^p-convolution operators on a free group*, pp. 333–343 in: *Harmonic Analysis, Proceedings, Cortona, Italy, 1982*, Lecture Notes in Math. 992, Springer, Berlin 1983.

[Py–Sz] T. Pytlik and R. Szwarc, *An analytic family of uniformly bounded representations of free groups*, Acta Math. 157 (1986), pp. 287–309.

FACHBEREICH 9 MATHEMATIK
UNIVERSITÄT DES SAARLANDES
D-6600 SAARBRÜCKEN, FEDERAL REPUBLIC OF GERMANY

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