COLLOQUIUM MATHEMATICUM

XVI DÉDIÉ À M. FRANCISZEK LEJA 1967

MEAN p-VALENT FUNCTIONS WITH GAPS

BY

W. K. HAYMAN (LONDON)

1. Introduction and statement of results. Suppose that

$$f(z) = \sum_{\nu=0}^{\infty} a_{n_\nu} z^{n_\nu} \tag{1.1}$$

is mean p-valent (1) in $|z| < 1$ and that

$$n_{\nu+1} - n_\nu \geqslant C, \qquad \nu \geqslant \nu_0, \tag{1.2}$$

where C is an integer greater than one. (The case $C=1$ is classical). Our aim is to investigate the effect of the gaps (1.1) and (1.2) on the growth of $f(z)$ and on its coefficients a_n. We shall prove the following

THEOREM 1. *Suppose that $f(z)$, given by (1.1), is mean p-valent in $|z| < 1$ and that (1.2) holds. Then*

$$M(r, f) < A_1(p, C, \nu_0)\mu_p(1-r)^{-2p/C}, \qquad 0 < r < 1, \tag{1.3}$$

and hence we have for $n \geqslant 1$

$$|a_n| < A_2(p, C, \nu_0)\mu_p n^{2p/C-1}, \qquad C < 4p, \tag{1.4}$$

$$|a_n| < A_3(p, C, \nu_0)\mu_p n^{-1/2}\log n, \qquad C = 4p. \tag{1.5}$$

If $C > 4p$,

$$|a_n| < A_4(p, C, \nu_0)\mu_p n^{-1/2}, \tag{1.6}$$

and

$$|a_n| = o(n^{-1/2}) \quad \textit{as} \quad n \to \infty. \tag{1.7}$$

Here

$$\mu_p = \max_{0 \leqslant n \leqslant p} |a_n|, \qquad M(r, f) = \max_{|z|=r} |f(z)|,$$

(1) In this paper mean p-valent denotes a really mean p-valent in the sense of [3].

and $A_j(p, C, \nu_0)$ denotes a particular constant depending on p, C, ν_0 only. In addition $A(\alpha, \beta, \gamma, \ldots)$ will denote as usual constants depending on $(\alpha, \beta, \gamma, \ldots)$ only, not necessarily the same each time. Theorem 1 is classical if $n_\nu = C\nu + B$, where B is a constant (see e.g. [3], Chapters II and III).

Inequalities (1.3), (1.4), (1.6) and (1.7) all give the correct orders of magnitude. In fact, if C is any positive integer and $p > 0$, the functions

$$f(z) = (1-z^C)^{-2p/C} = \sum_{\nu=0}^{\infty} a_{C\nu} z^{C\nu}$$

satisfy (1.1) with $n_\nu = C\nu$, and are p-valent in $|z| < 1$ if p/C is an integer and mean p-valent otherwise. Also

$$M(r, f) = (1-r^C)^{-2p/C}$$

and

$$a_{C\nu} \sim \frac{n^{2p/C-1}}{\Gamma(2p/C)} \quad \text{as} \quad n \to \infty,$$

so that the orders of magnitude in (1.3) and (1.4) cannot be sharpened. The deductions (1.6) and (1.7) from (1.3) hold for any mean p-valent function and are due to Pommerenke [6]. These inequalities are also sharp. In fact, if $\{a_n\}$ is any sequence of positive numbers for $n \geqslant 1$, such that

$$\sum_1^\infty a_n \leqslant 1, \quad \sum_1^\infty n a_n^2 \leqslant 1, \tag{1.8}$$

then

$$f(z) = a_0 + \sum_1^\infty a_n z^n \tag{1.9}$$

is mean p-valent in $|z| < 1$, provided that $a_0 > 1 + p^{-1/2}$ [6]. For a fixed m we may choose $a_n = n^{-1/2}$, $n = m$, $a_n = 0$ otherwise, so that (1.6) cannot be sharpened. Also given any sequence ε_n, such that

$$\varepsilon_n \to 0 \quad \text{as} \quad n \to \infty,$$

we can clearly find a sequence n_ν of positive integers satisfying (1.2) and such that

$$\sum_1^\infty \varepsilon_{n_\nu} \leqslant 1, \quad \sum_1^\infty \varepsilon_{n_\nu}^2 \leqslant 1.$$

We then set $a_0 = 2 + p^{-1/2}$,

$$a_n = \varepsilon_n / n^{1/2}, \quad n = n_\nu, \; a_n = 0, \; n \neq n_\nu, \; n \geqslant 1,$$

and note that (1.8) holds so that $f(z)$ given by (1.9) is mean p-valent. Since ε_n may tend to zero as slowly as we please, the order of magnitude in (1.7) cannot be sharpened. It is not known whether (1.5) is sharp.

If p is an integer, the functions

$$f(z) = z(1-z^C)^{-2p/C}$$

are p-valent, i.e. assume no value more than p times and show that even in this case (1.3) and (1.4) cannot be sharpened. It is not known whether (1.6) and (1.7) remain sharp for p-valent functions. However, some examples of Littlewood [5], show that, if $p = 1$ and $f(z)$ is univalent, (1.4) does not in general remain true if C is large even for the sequence $n_\nu = C\nu+1$. In the second half of the paper we investigate those functions which have maximal growth subject to the hypotheses of Theorem 1.

2. Proof of Theorem 1. Inequalities (1.4) to (1.7) are immediate consequences of (1.3) for mean p-valent functions by theorems of Spencer ([8] see also [3]) and Pommerenke [6]. It is thus only necessary to prove (1.3). We shall see that (1.3) also follows fairly simply from known results. Among these the following Theorem of Ingham ([4], Theorem 1) plays a fundamental role.

LEMMA 1. *Suppose that $g(\theta)$, defined in $[0, 2\pi]$, has a uniformly convergent Fourier expansion*

$$g(\theta) = \sum_{-\infty}^{\infty} b_{n_\nu} e^{in_\nu\theta},$$

where $n_{\nu+1}-n_\nu \geqslant C \geqslant 1$. Then given $\varepsilon > 0$, $0 \leqslant \theta_1 < 2\pi$, we have

$$\int_0^{2\pi} |g(\theta)|^2 d\theta < CA(\varepsilon) \int_{\theta_1}^{\theta_1+2\frac{\pi+\varepsilon}{C}} |g(\theta)|^2 d\theta.$$

We now write

$$(2.1)\qquad P(z) = \sum_{\nu=0}^{\nu_0-1} a_{n_\nu} z^{n_\nu}, \qquad g(z) = \sum_{\nu=\nu_0}^{\infty} a_{n_\nu} z^{n_\nu},$$

so that $f(z) = P(z)+g(z)$ in (1.1).

We have by Lemma 1, for $0 < t < 1$, $0 \leqslant \theta_1 < 2\pi$, $\varepsilon > 0$ and $\theta_2 \geqslant \theta_1+2(\pi+\varepsilon)/C$

$$(2.2)\qquad \int_0^{2\pi} |g'(te^{i\theta})|^2 d\theta \leqslant CA(\varepsilon) \int_{\theta_1}^{\theta_2} |g'(te^{i\theta})|^2 d\theta.$$

We write

$$S(r, \theta_1, \theta_2, g) = \int_0^r \int_{\theta_1}^{\theta_2} |g'(te^{i\theta})|^2 t dt d\theta,$$

$$S(r, g) = S(r, 0, 2\pi, g),$$

and deduce that for $\theta_2 \geqslant \theta_1 + 2\left(\frac{\pi+\varepsilon}{C}\right)$

$$S(r, g) < CA(\varepsilon) S(r, \theta_1, \theta_2, g). \tag{2.3}$$

We have next

Lemma 2. *Suppose that $f(z)$ is regular in $|z| < 1$, that α and K are positive constants and that*

$$M(r_1, f) < K(1-r_1)^{-\alpha}, \quad M(r_2, f) \geqslant K(1-r_2)^{-\alpha}, \tag{2.4}$$

where $0 < r_1 < r_2 < 1$. Then there exists r such that $r_1 < r \leqslant r_2$ and

$$M(r, f) \geqslant K(1-r)^{-\alpha}, \tag{2.5}$$

$$S(r', f) \geqslant \frac{\pi}{4} \alpha^2 M(r, f)^2 \geqslant \frac{\pi}{4} \alpha^2 K^2 (1-r)^{-2\alpha}, \tag{2.6}$$

where $r' = \frac{1}{2}(1+r)$.

We write $M(r) = M(r, f)$ and denote by $M'(r)$ the right derivative of $M(r)$. Suppose that

$$\frac{M'(t)}{M(t)} \leqslant \frac{\alpha}{(1-t)}, \quad r \leqslant t \leqslant r_2. \tag{2.7}$$

Then we have by integration

$$\log M(r_2) - \log M(r) \leqslant \alpha \log\left(\frac{1-r}{1-r_2}\right),$$

so that

$$(1-r)^\alpha M(r) \geqslant (1-r_2)^\alpha M(r_2) \geqslant K. \tag{2.8}$$

This contradicts (2.3) if $r = r_1$ and so (2.7) is certainly false for $r = r_1$. If

$$\frac{M'(r_2)}{M(r_2)} \geqslant \frac{\alpha}{(1-r_2)},$$

we set $r = r_2$. If not, we set r equal to the lower bound of all numbers such that (2.7) holds. It then follows that $r_1 < r \leqslant r_2$ and further that

$$\frac{M'(r)}{M(r)} \geqslant \frac{\alpha}{1-r}. \tag{2.9}$$

In fact, $M(r)$ can have isolated jump increases, but by our definition $M'(\varrho)$ is continuous on the right. If (2.9) were false, we could replace r by a slightly smaller number. Also in view of (2.8) we have (2.5).

We proceed to prove (2.6). It follows from a known result (see e.g. [2]) that there always exists a point $z_0 = re^{i\theta}$, such that

$$|f(z_0)| = M(r, f), \qquad z_0 \frac{f'(z_0)}{f(z_0)} = r \frac{M'(r)}{M(r)}.$$

Hence

$$|f'(z_0)| = M'(r).$$

Now if $r' = \frac{1}{2}(1+r)$ it follows that

$$S(r', f) \geqslant \iint\limits_{|z-z_0|\leqslant r'-r} |f'(z)|^2 dxdy \geqslant \pi(r'-r)^2 |f'(z_0)|^2,$$

since $|f'(z)|^2$ is subharmonic. This gives, in view of (2.8) and (2.9),

$$S(r', f) \geqslant \frac{\pi}{4}(1-r)^2 M'(r)^2 \geqslant \frac{\pi}{4}\alpha^2 M(r, f)^2 \geqslant \frac{\pi}{4}\alpha^2 K^2 (1-r)^{-2\alpha},$$

which is (2.6). This completes the proof of Lemma 2.

We next need a lemma due to Pommerenke [7].

LEMMA 3. *If*

$$f(z) = \sum_1^\infty a_n z^n$$

is mean p-valent in $|z| < 1$, and $\delta > 0$, then there exists a positive integer k, and numbers $C_0, C_1, C_2, \ldots, C_k$ possibly depending on n but bounded above by constants depending on p, δ only such that

$$\left| a_n + \sum_{t=1}^{k} C_t a_{n-t} \right| < C_0 n^{-1/2+\delta} \mu_p. \tag{2.10}$$

We deduce

LEMMA 4. *With the hypotheses of Theorem 1, and given $\delta > 0$, we have*

$$|a_n| < A(p, \delta, \nu_0) n^{-1/2+\delta} \mu_p, \qquad n \leqslant n_{\nu_0}.$$

Since $a_n = 0$, except when $n = n_\nu$, it is sufficient to consider $n = n_\nu$, with $\nu \leqslant \nu_0$. We prove our result by induction on ν_0. Suppose that it is proved for $\nu_0 < m$. We proceed to prove it for $\nu_0 = m$. We set

$$N = n_{\nu_0} = n_m$$

and use Lemma 3 with $n = N$. The terms a_{n-t} which appear in (2.10) and are different from zero are of the form a_{n_ν}, with $\nu < m$, and so Lemma 3 applies to these. We deduce

$$|a_n| < A(p, \delta)\mu_p n^{-1/2+\delta} + \sum_{t=1}^{k} \mu_p A(p, \delta, \nu_0 - t, t)(n-t)^{-1/2+\delta}$$
$$< \mu_p A(p, \delta, \nu_0) n^{-1/2+\delta}.$$

This proves Lemma 4.

We have next

LEMMA 5. *If $f(z)$ satisfies the hypotheses of Theorem 1 and $P(z)$ is given by (2.1), then given $\alpha > 0$, we have*

$$S(r, P) < A_1(p, \alpha, \nu_0)\mu_p^2(1-r)^{-2\alpha}, \qquad 0 < r < 1.$$

We have by Lemma 4

$$S(r, P) = \pi \sum_{\nu=1}^{\nu_0} n_\nu |a_{n_\nu}|^2 r^{2n_\nu} < A(p, \alpha, \nu_0)\mu_p^2 \sum_{\nu=1}^{\nu_0} n_\nu^{2\alpha} r^{2n_\nu}$$
$$< A(p, \alpha, \nu_0)\mu_p^2 \sup_{n=1 \text{ to } \infty} \{n^{2\alpha} r^{2n}\}.$$

Also

$$\frac{(n+1)^{2\alpha} r^{2(n+1)}}{n^{2\alpha} r^{2n}} = \left(1 + \frac{1}{n}\right)^{2\alpha} r^2 < 1,$$

if $r < (1+1/n)^{-\alpha}$, $n > (r^{-1/\alpha} - 1)^{-1}$. Thus

$$\sup n^{2\alpha} r^{2n} \leqslant [(r^{-1/\alpha} - 1)^{-1} + 1]^{2\alpha} < A(\alpha)(1-r)^{-2\alpha}.$$

This proves Lemma 5.

We can now prove

LEMMA 6. *Suppose that $f(z)$ satisfies the hypotheses of Theorem 1 and that (2.4) holds with $\alpha = 2p/C$ and some constant*

$$K > A_2(\nu_0, p, C, \varepsilon)\mu_p. \tag{2.11}$$

Let r, r' satisfy the conclusions of Lemma 2. Then if $\varepsilon > 0$, $0 \leqslant \theta_1 < 2\pi$, there exists $z = \varrho e^{i\theta}$, with $0 \leqslant \varrho \leqslant r'$, $\theta_1 \leqslant \theta \leqslant \theta_1 + 2(\pi + \varepsilon)/C$, such that

$$|f(z)| > \frac{K}{4}\left[\frac{p}{C^3 A(\varepsilon)}\right]^{1/2} (1-r)^{-\alpha}, \tag{2.12}$$

where $A(\varepsilon)$ is the constant of (2.3).

We have by (2.5) and (2.6)

$$S(r',f) \geqslant \frac{\pi}{4} K^2 a^2 (1-r)^{-2a},$$

while by Lemma 5 with $a = 2p/C$ we have

$$S(r',P) \leqslant A_1 \mu_p^2 (1-r)^{-2a}, \tag{2.13}$$

where A_1 depends on r_0, p, C only. We have for any two functions φ_1 and φ_2

$$S(r,\varphi_1+\varphi_2) = \iint\limits_{|z|<r} |\varphi_1+\varphi_2|^2 dxdy \geqslant S(r,\varphi_1)+S(r,\varphi_2)-2[S(r,\varphi_1)S(r,\varphi_2)]^{1/2}$$

by Schwarz's inequality. Thus setting $\varphi_1 = f$, $\varphi_2 = -P$, we have

$$S(r',g) \geqslant S(r',f)+S(r',P)-2[S(r',f)S(r',P)]^{1/2} \geqslant \frac{1}{4} S(r',f)$$

provided that

$$S(r',P) \leqslant \frac{1}{16} S(r',f),$$

and this is true provided that

$$\frac{\pi^2}{4} K^2 a^2 \geqslant 16 A_1 \mu_p^2 \tag{2.14}$$

which we assume. We now apply (2.3) and deduce that we have, with $\theta_2 = \theta_1 + 2(\pi+\varepsilon)/C$,

$$S(r',\theta_1,\theta_2,g) > \big(A(\varepsilon)C\big)^{-1} S(r',g) \geqslant \frac{S(r',f)}{4CA(\varepsilon)} \geqslant \frac{\pi a^2 K^2}{16CA(\varepsilon)(1-r)^{2a}}.$$

Suppose now that K is so large that

$$\frac{\pi^2 a^2 K^2}{16CA(\varepsilon)} \geqslant 16 A_1 \mu_p^2. \tag{2.15}$$

Then we deduce from Lemma 5 that

$$S(r',\theta_1,\theta_2,P) \leqslant S(r',P) \leqslant \frac{1}{16} S(r',\theta_1,\theta_2,g).$$

Since $f = P+g$, this gives just as before that

$$S(r',\theta_1,\theta_2,f) \geqslant \frac{1}{4} S(r',\theta_1,\theta_2,g) \geqslant \frac{\pi a^2 K^2}{64CA(\varepsilon)(1-r)^{2a}}.$$

Let $M = \sup|f(z)|$ in the sector E of values $z = te^{i\theta}$ for which $\theta_1 \leqslant \theta \leqslant \theta_2$, $0 \leqslant t \leqslant r'$. Since $f(z)$ is mean p-valent in E, the area of the image of this set by $f(z)$ is at most $\pi p M^2$, so that

$$p\pi M^2 \geqslant S(r', \theta_1, \theta_2, f) \geqslant \frac{\pi \alpha^2 K^2}{64CA(\varepsilon)(1-r)^{2\alpha}}.$$

Thus we can find $z = te^{i\theta}$ in E such that

$$|f(z)| = M \geqslant \frac{\alpha K}{8\sqrt{[pCA(\varepsilon)]}}(1-r)^{-\alpha} = \frac{K}{4}\left(\frac{p}{C^3 A(\varepsilon)}\right)^{1/2}(1-r)^{-\alpha}.$$

This proves Lemma 6 provided that K satisfies (2.14) and (2.15), i.e. provided that

$$K > A(\nu_0, p, C, \varepsilon)\mu_p,$$

as required.

3. Lemma 6 tells us that there exist at least C points z in $|z| \leqslant r'$ no two of which are too close together such that (2.12) holds. This leads to a contradiction if K is too large. We have more precisely

LEMMA 7. *With the hypotheses of Lemma 6, we can find points* $z_\nu = \varrho_\nu e^{i\theta_\nu}$, $\nu = 0$ *to* $C-1$, *such that*

$$1-4^{-p}\varepsilon/(\pi C) \leqslant \varrho_\nu \leqslant r' \quad \text{and} \quad |\theta_\mu - \theta_\nu| \geqslant \frac{2\varepsilon}{C} \bmod 2\pi,\ 0 \leqslant \mu < \nu \leqslant C-1,$$

where $\varepsilon = \pi/[2(C-1)]$, *and*

(3.1) $$|f(z_\nu)| > KA(p, C)(1-r')^{-\alpha}.$$

We choose $z_0 = r'e^{i\theta_0}$, such that

$$|f(z_0)| = M(r', f) \geqslant M(r, f) \geqslant K(1-r)^{-\alpha} = 2^{-\alpha}K(1-r')^{-\alpha}$$

by (2.5). We then find $z = r_\nu e^{i\theta_\nu}$, $\nu = 1$ to $C-1$, satisfying inequality (2.12) in each of the ranges

$$\theta_0 + \frac{2\pi(\nu-1)}{C-1} + \frac{\varepsilon}{C} < \theta_\nu < \theta_0 + \frac{2\pi\nu}{C-1} - \frac{\varepsilon}{C},$$

where

$$\varepsilon = \frac{\pi}{2(C-1)}.$$

The length of the interval in which θ_ν is allowed to lie is

$$\frac{2\pi}{(C-1)} - \frac{2\varepsilon}{C} = 2\pi\left[\frac{1}{C-1} - \frac{1}{2C(C-1)}\right] = 2\pi\left[\frac{1}{C} + \frac{1}{2C(C-1)}\right] = \frac{2(\pi+\varepsilon)}{C},$$

so that Lemma 6 is applicable. Also if we define $\theta_C = \theta_0+2\pi$, we have

$$\theta_\nu \geqslant \theta_{\nu-1}+\frac{2\varepsilon}{C}, \qquad \nu = 1 \text{ to } C,$$

which is the required inequality for the points θ_ν. Also since $\alpha = 2p/C$ and $\varepsilon = \pi/[2(C-1)]$, (2.12) becomes

$$|f(z_\nu)| > \frac{K\sqrt{p}}{A(C)}(1-r)^{-\alpha} = \frac{K\sqrt{p\cdot 2^{-2p/C}}}{A(C)}(1-r')^{-\alpha},$$

which gives (3.1).

By (2.12) we have $|z_\nu| \leqslant r'$. To make $|z_\nu| \geqslant 1-4^{-p}\varepsilon/\pi C$ it is sufficient to choose K so big that

$$M\left(1-\frac{4^{-p}\varepsilon}{\pi C}, f\right) < KA(C, p).$$

Since $f(z)$ is mean p-valent in $|z| < 1$

$$M\left(1-\frac{4^{-p}\varepsilon}{\pi C}, f\right) < A(p, C)\mu_p\varepsilon^{-2p} = A(p, C)\mu_p,$$

so that we can achieve this by increasing if necessary the constant on the right-hand side of (2.11).

We quote one final result ([3], Theorem 2.6):

LEMMA 8. *Suppose that $f(z)$ is mean p-valent in the union of the disjoint disks $|z-z_\nu'| < r_\nu$, $\nu = 1$ to C, and that $f(z) \neq 0$ in $|z-z_\nu'| < \frac{1}{2}r_\nu$. Suppose also that*

$$|f(z_\nu')| \leqslant R_1, \qquad |f(z_\nu)| \geqslant R_2,$$

where $\delta_\nu = (r_\nu - |z_\nu - z_\nu'|)/r_\nu > 0$, $R_2 > eR_1$. Then

$$\sum_{\nu=1}^{C}\left[\log\left(\frac{A(p)}{\delta_\nu}\right)\right]^{-1} \leqslant \frac{2p}{\log(R_2/R_1)-1}.$$

We can now complete our proof [2]. Since $f(z)$ is mean p-valent in $|z| < 1$, $f(z)$ has $q \leqslant p$ zeros there. Hence $f(z) \neq 0$ in at least one of the annuli

$$1-\frac{4^{-t}2\varepsilon}{\pi C} < |z| < 1-\frac{4^{-(t+1)}2\varepsilon}{\pi C}, \qquad t = 0 \text{ to } q.$$

We choose such an annulus, set

$$\varrho = 1-4^{-t}\frac{\varepsilon}{\pi C} = 1-r_0$$

---

[2] The argument is almost identical with that for [3], Theorem 3.7.

and note that $f(z)$ has no zeros in the annulus $1-2r_0 < |z| < 1-\frac{1}{2}r_0$. We set $z_\nu' = \varrho e^{i\theta}$, $z_\nu = \varrho_\nu e^{i\theta_\nu}$, $r_\nu = r_0$ in Lemma 8, and note that by hypothesis

$$\varrho_\nu \geqslant 1 - \frac{4^{-p}\varepsilon}{\pi C} \geqslant \varrho .$$

Also if $\mu \neq \nu$,

$$|z_\mu' - z_\nu'| \geqslant 2\varrho \left| \sin\left(\frac{\theta_\mu - \theta_\nu}{2}\right)\right| \geqslant \varrho \frac{4\varepsilon}{\pi C} \geqslant \frac{2\varepsilon}{\pi C} > 2r_0 .$$

Thus the disks $|z - z_\mu'| < r_0$ lie in $|z| < 1$ and are disjoint and so we can apply Lemma 8. Finally

$$\delta_\nu = \frac{r_0 - (\varrho_\nu - \varrho)}{r_0} = \frac{1-\varrho_\nu}{r_0} \geqslant \frac{1-r'}{r_0} .$$

We set

$$R_1 = M(\varrho, f) < A(p)\mu_p(1-\varrho)^{-2p} < A(p)\mu_p C^{4p}$$

by a classical result of Spencer (see e.g. [3], Theorem 2.5). We also put

$$R_2 = \inf_{\nu = 0 \text{ to } C-1} |f(z_\nu)| \geqslant KA(p, C)(1-r')^{-2p/C} .$$

We have by Lemma 8

$$\text{either} \quad R_2 < eR_1 \quad \text{or} \quad \sum_{\nu=1}^{C} \left[\log \frac{A(p)}{\delta_\nu}\right]^{-1} < \frac{2p}{\log(R_2/R_1) - 1},$$

i.e.

$$\log\left(\frac{R_2}{R_1}\right) < 1 + \frac{2p}{C} \log \frac{A(p) r_0}{(1-r')},$$

$$R_2 < eR_1 \left[\frac{A(p) r_0}{(1-r')}\right] 2p/C,$$

and hence

$$R_2 < A(p, C)\mu_p(1-r')^{-2p/C} .$$

Thus this inequality holds in any case. We deduce

$$KA(p, C)(1-r')^{-2p/C} < A(p, C)\mu_p(1-r')^{-2p/C},$$

which gives $K \leqslant A(p, C)\mu_p$.

Thus (2.4) leads to a contradiction if (2.11) is satisfied and $K > A(p, C)$. Since for arbitrarily large K, we can always satisfy

$$M(r_1, f) < K(1-r_1)^{-\alpha},$$

e.g. with $r_1 = \frac{1}{2}$, it follows that the contradiction must arise from

$$M(r_2, f) \geqslant K(1-r_2)^{-a},$$

which must therefore be false for $K \geqslant A(p, C, \nu_0)\mu_p$. Thus

$$M(r, f) < A(p, C, \nu_0)\mu_p(1-r)^{-2p/C}, \quad 0 < r < 1,$$

as required and the proof of Theorem 1 is complete.

4. Suppose now that the hypotheses of Theorem 1 hold and set

$$\beta = \overline{\lim_{r\to 1}}\,(1-r)^{2p/C} M(r, f).$$

If $\beta = 0$, it follows from classical arguments ([3], pp. 46 and 105) that (1.4) can be sharpened to

$$|a_n| = o(n^{2p/C-1}) \quad \text{as} \quad n \to \infty,$$

if $C < 4p$. We have no further conclusions in this case and confine ourselves in what follows to the hypothesis

$$\beta > 0. \tag{4.1}$$

In this case we are able to apply a series of rather deep regularity theorems recently obtained by Eke [1]. We start by proving that the hypotheses of Eke's Theorems hold. We have

LEMMA 9. *Suppose that $f(z)$ satisfies the hypotheses of Theorem 1 and in addition (4.1). Then there exists a sequence r_k, with the following properties:*

$$0 < r_k < 1, \quad k = 1, 2, \ldots,$$
$$r_k \to 1, \quad \textit{as} \quad k \to \infty.$$

Further for each k there exist C points $z_{\nu,k}$, $\nu = 0$ to $C-1$, such that

$$|z_{\nu,k}| = r_k, \quad \nu = 0 \text{ to } C-1,$$
$$|z_{\nu,k} - z_{\mu,k}| \geqslant \delta > 0, \quad 0 \leqslant \mu < \nu \leqslant C-1,$$

where δ is a positive constant independent of μ, ν and k, and finally

$$|f(z_{\nu,k})| > \beta'(1-r_k)^{-2p/C}, \quad k = 1 \text{ to } \infty, \nu = 1 \text{ to } C,$$

where β' is a constant independent of ν and k.

We note first that

$$\beta_1 = \overline{\lim_{r\to 1}}\,(1-r)^{(2p/C+1)} M(r, f') \geqslant 2p\beta/C.$$

For if this is false we have for $|z| = r > r_0$

$$|f'(z)| \leqslant (\beta_1 + \varepsilon)(1-r)^{-2p/C-1}.$$

Integrating along a radius this gives for $r > r_0$

$$M(r, f) < M(r_0, f) + (\beta_1+\varepsilon)\frac{C}{2p}(1-r)^{-2p/C},$$

which yields a contradiction if $\beta_1 C/(2p) < \beta$. Next we note that

$$\beta_2 = \overline{\lim_{r\to 1}}\,(1-r)^{4p/C} S(r, f') \geqslant A(p, C)\beta^2. \tag{4.2}$$

In fact, suppose that $\varepsilon > 0$ and let r be chosen so that $1-\varepsilon < r < 1$ and such that there exists z_0 with $|z_0| = r$ and

$$|f'(z_0)| > (\beta_1-\varepsilon)(1-r)^{-(2p/C+1)}.$$

Then if $r' = \frac{1}{2}(1+r)$, we have as in the proof of Lemma 2

$$S(r', f) \geqslant \iint_{|z-z_0|<\frac{1}{2}(1-r)} |f'(z)|^2 dxdy \geqslant \frac{\pi}{4}(1-r)^2 |f'(z_0)|^2$$

$$\geqslant \frac{\pi}{4}(\beta_1-\varepsilon)^2(1-r)^{-4p/C} = \frac{\pi}{4}2^{-4p/C}(\beta_1-\varepsilon)^2(1-r')^{-4p/C},$$

and since r' can be chosen as near 1 as we please we deduce (4.2).

We now choose r as near 1 as we please such that

$$S(r, g) > \frac{1}{2}\beta_2(1-r)^{-4p/C},$$

where $g(z)$ is derived from $f(z)$ as in (2.1). This is possible since $P(z)$ is a polynomial, so that $f'(z)-g'(z)$ is bounded. It then follows from (2.3) that

$$S(r, \theta_1, \theta_2, g) > \frac{A(\varepsilon)}{2C}\beta_2(1-r)^{-4p/C},$$

provided that $\theta_2 > \theta_1 + 2(\pi+\varepsilon)/C$, and hence also that

$$S(r, \theta_1, \theta_2, f) \geqslant \frac{A(\varepsilon)}{3C}\beta_2(1-r)^{-4p/C}$$

with the same hypotheses. Since $f(z)$ is mean p-valent, it follows that

$$S(r, \theta_1, \theta_2, f) \leqslant \pi p M^2$$

where M is the maximum of $|f(te^{i\theta})|$ for $0 < t < r$, $\theta_1 < \theta < \theta_2$. Hence we can find $te^{i\theta}$ such that

$$|f(te^{i\theta})| > \left(\frac{A(\varepsilon)\beta_2}{3\pi p C}\right)^{1/2}(1-r)^{-2p/C}.$$

On the other hand, we have by hypothesis if r and so t is sufficiently near 1,

$$|f(te^{i\theta})| < 2\beta(1-t)^{-2p/C}, \tag{4.3}$$

so that

$$(1-t) > K(1-r), \tag{4.4}$$

where K is a positive constant depending on β, p, C and ε only. Also we may suppose that r and hence t are so near 1 that all the zeros of $f(z)$, of which there are at most p, lie in $|z| < 2t-1$. Then

$$\varphi(z) = f[te^{i\theta}+(1-t)z]$$

is mean p-valent and non zero in $|z| < 1$ and so we deduce that ([3], p. 23)

$$|\varphi(z)| > A(p)(1-|z|)^{2p}|\varphi(0)|, \qquad |z| < 1.$$

We apply this result with $te^{i\theta}+(1-t)z = re^{i\theta}$, so that

$$|z| = (r-t)/(1-t), \qquad (1-|z|) = (1-r)/(1-t),$$

and in view of (4.3) and (4.4) we deduce that

$$|f(re^{i\theta})| > A(p, C, \varepsilon)\beta(1-r)^{-2p/C}, \tag{4.5}$$

for some θ in each range $\theta_1 < \theta < \theta_2$, provided that

$$\theta_2 > \theta_1+2(\pi+\varepsilon)/C,$$

and this is true for some r arbitrarily near 1. We choose again θ_0' so that

$$|f(re^{i\theta_0'})| = M(r, f)$$

and then choose a value $\theta = \theta_\nu'$ to satisfy (4.5) in each of the ranges

$$\theta_0+\frac{2\pi(\nu-1)}{C-1}+\frac{\varepsilon}{C} < \theta_\nu' < \theta_0+\frac{2\pi\nu}{C-1}-\frac{\varepsilon}{C}, \qquad \nu = 1 \text{ to } C-1,$$

where $\varepsilon = \pi/[2(C-1)]$. Then if $z_\nu = re^{i\theta_\nu'}$, $\nu = 0$ to $C-1$, we obtain the conclusion of Lemma 9. We have only to let r tend to 1 through a suitable sequence of values r_k and write $z_{\nu,k}$ for the corresponding value of z_ν.

Using merely the hypothesis that $f(z)$ is mean p-valent in $|z| < 1$, and satisfies the conclusions of Lemma 9 Eke [1] deduces a remarkable series of conclusions which we summarize as follows:

LEMMA 10. *If*

$$f(z) = \sum_0^\infty a_n z^n$$

is mean p-valent in $|z| < 1$ *and satisfies the conclusions of Lemma 9, then there exist k rays* $\arg z = \theta_\nu$, *where* $\theta_0 < \theta_1 < \theta_2 < \ldots < \theta_c = \theta_0 + 2\pi$ *with the following properties:*

(i) $$\log|f(re^{i\theta_\nu})| = \frac{2p}{C}\log\frac{1}{1-r} + o\left(\log\frac{1}{1-r}\right)^{1/2}, \qquad \nu = 0 \text{ to } C-1,$$

as $r \to 1$,

(ii) *Further if* $r_\nu = r_\nu(R)$ *is so chosen that* $|f(r_\nu e^{i\theta_\nu})| = R$, *then*

$$R^{C^2}\prod_{\nu=0}^{C-1}(1-r_\nu)^{2p} \to \beta_3$$

as $R \to \infty$, *where* $0 < \beta_3 < \infty$.

(iii) *If we write*

$$a_\nu(n) = n^{-2p/C} f\left[\left(1-\frac{1}{n}\right)e^{i\theta_\nu}\right],$$

then we have

$$f(re^{i\theta}) = [1+o(1)]\,a_\nu(n)\,[1-re^{i(\theta-\theta_\nu)}]^{-2p/C}$$

as $n \to \infty$, *uniformly provided that* $n(1-r)$ *is bounded above and below and* $n|\theta-\theta_\nu|$ *is bounded above. Also*

$$\frac{f'(re^{i\theta})}{f(re^{i\theta})} \sim \frac{2p}{C[1-re^{i(\theta-\theta_\nu)}]},$$

while $r \to 1$ *and* $|\theta-\theta_\nu| = O(1-r)$.

(iv) *If, for* $\nu = 1$ *to* C, $|\theta-\theta_\nu| \geqslant \delta > 0$, *where* δ *is a fixed positive constant, then*

$$\log|f(re^{i\theta})| = o\left(\log\frac{1}{1-r}\right)^{1/2},$$

uniformly as $r \to 1$.

(v) *If in addition* $4p > C$, *then we have as* $n \to \infty$

$$a_n = \sum_{\nu=0}^{C-1} a_\nu(n)\,\frac{n^{2p/C-1}}{\Gamma[2p/C]}\,e^{-in\theta_\nu} + o[\alpha(n)\,n^{2p/C-1}],$$

where

$$\alpha(n) = \sup_{\nu=0 \text{ to } C-1} a_\nu(n).$$

Thus the functions $f(z)$ satisfying the hypotheses of Theorem 1 and in addition (4.1) satisfy all the conclusions of Lemma 10. However, in this special situation we can prove a little more.

THEOREM 2. *Suppose that $f(z)$ satisfies the hypotheses of Theorem 1 and in addition (4.1) holds. Then $f(z)$ satisfies the conclusions of Lemma 10. In addition we have with the notation of that Lemma*

(vi) $\theta_\nu = \theta_0 + \frac{2\pi\nu}{C}$, $\nu = 0$ *to* $C-1$,

(vii) *We have, as* $r \to 1$,

$$\log|f(re^{i\theta_\nu})| = \frac{2p}{C}\log\frac{1}{1-r} + O(1), \qquad \nu = 0 \text{ to } C-1,$$

If further $p > 4C$ *we can strengthen this to*

(vii′) $|f(re^{i\theta_\nu})| = |f(re^{i\theta_0})| + O(1) \sim \beta(1-r)^{-2p/C}$, *as* $r \to 1$.

Also

(viii) *There exists an integer B such that, for all sufficiently large ν, $n_\nu = C\nu + B$ in (1.1) and*

$$|a_{n_\nu}| \sim \frac{\beta C n_\nu^{2p/C-1}}{\Gamma(2p/C)}, \qquad as \qquad \nu \to \infty.$$

5. Proof of Theorem 2. We start by proving (vi). Suppose that the result is false. Then there exists ν such that $0 \leqslant \nu < C$ and

$$\theta_{\nu+1} - \theta_\nu > \frac{2\pi}{C}.$$

We define ε such that

$$\theta_{\nu+1} - \theta_\nu = \frac{2\pi + 5\varepsilon}{C}$$

and set

$$\theta'_\nu = \theta_\nu + \frac{\varepsilon}{C}, \qquad \theta'_{\nu+1} = \theta_{\nu+1} - \frac{\varepsilon}{C}.$$

It then follows from Lemma 10 (iv) that

$$\log|f(re^{i\theta})| = O\left\{\log\frac{1}{1-r}\right\}^{1/2}$$

uniformly for $\theta'_\nu \leqslant \theta \leqslant \theta'_{\nu+1}$. This however contradicts (4.5). Thus we have (vi).

We next prove (vii). Let θ'_ν be a fixed number such that $\theta_\nu < \theta'_\nu < \theta_{\nu+1}$, $\nu = 0$ to $C-1$, and $\theta'_C = \theta'_0 + 2\pi$. Then if $r_n = 1 - 1/n$, it follows easily from (iii) and (iv) that

$$S[r_n, \theta'_{\nu-1}, \theta'_\nu, f] \geqslant [1+o(1)]|a_\nu(n)|^2 S[r_n, (1-z)^{-2p/C}]. \tag{5.1}$$

The method of Eke [1] also yields

$$(5.2) \qquad S[r_n, f) = [1+o(1)]\Big\{\sum_{\nu=0}^{C-1} |a_\nu(n)|^2\Big\} S[r_n, (1-z)^{-2p/C}].$$

If $P(z)$ and $g(z)$ are again defined as in (2.1), the analogous asymptotic relations hold for $g(z) = f(z)-P(z)$, since $P(z)$ is a polynomial. We now choose for a fixed θ_ν

$$\theta'_{\nu-1} = \theta_\nu - \frac{3\pi}{2C}, \qquad \theta'_\nu = \theta_\nu + \frac{3\pi}{2C},$$

which is equivalent to choosing $\varepsilon = \pi/2$, when applying (2.3) with $\theta'_{\nu-1}$, θ'_ν instead of θ_1, θ_2. We deduce from (2.3) that

$$S(r_n, f) < ACS(r_n, \theta'_{\nu-1}, \theta'_\nu, f)$$

where A is an absolute constant, i.e. for each fixed ν and $n > n_0$

$$\sum_{\nu=0}^{C-1} |a_\nu(n)^2| < 2AC|a_\nu(n)|^2,$$

in view of (5.2). This yields, for each fixed ν,

$$(5.3) \qquad |a(n)| \leqslant \sqrt{(2AC)}|a_\nu(n)|.$$

We now use Lemma 10 (ii) and set

$$R = (\beta_3 n^{2pC})^{1/C^2}.$$

Then if the r_ν are defined as in Lemma 10 (ii) it follows that

$$\inf_{\nu=0 \text{ to } C-1} (1-r_\nu) \leqslant \frac{1+o(1)}{n}, \qquad \sup_{\nu=0 \text{ to } C-1} (1-r_\nu) \geqslant \frac{1+o(1)}{n}.$$

It also follows from (iii) that $|f(re^{i\theta_\nu})|$ finally increases for each fixed ν. Thus we deduce that

$$\inf_{\nu=0 \text{ to } C-1} |f(r_n e^{i\theta_\nu})| \leqslant [1+o(1)]R,$$

$$\sup_{\nu=0 \text{ to } C-1} |f(r_n e^{i\theta_\nu})| \geqslant [1+o(1)]R.$$

This is equivalent to

$$\inf_{\nu=0 \text{ to } C-1} |a_\nu(n)| \leqslant [1+o(1)]\beta_3^{1/C^2},$$

$$|a(n)| \geqslant [1+o(1)]\beta_3^{1/C^2},$$

and in view of (5.3) we deduce that for large n

$$\frac{\beta_3^{1/C^2}}{\sqrt{(3AC)}} < |a_\nu(n)| \leqslant |a(n)| \leqslant \sqrt{(3AC)}\,\beta_3^{1/C^2}.$$

Since for $1-1/n \leqslant r \leqslant (1-1/(n+1))$ we have

$$|f(re^{i\theta_\nu})| \sim |f(r_n e^{i\theta_\nu})|,$$

we deduce that for each ν

$$\frac{\beta_3^{1/C^2}}{\sqrt{(3AC)}} \leqslant \underline{\lim}_{r\to 1}(1-r)^{2p/C}|f(re^{i\theta_\nu})| \leqslant \overline{\lim}_{r\to 1}(1-r)^{2p/C}|f(re^{i\theta_\nu})| \leqslant \sqrt{(3AC)}\beta_3^{1/C^2}.$$

This gives (vii).

We next suppose that $p > 4C$ and prove (viii). We can in this case make use of the asymptotic formula (v). By hypothesis $n_{\nu+1}-n_\nu \geqslant C$. To show that $n_\nu = C\nu+B$ finally it is enough to show that if n is large and Lemma 10 (v) holds with $\theta_\nu = \theta_0+2\pi\nu/C$, then it is not possible for the coefficients $a_n, a_{n+1}, \ldots, a_{n+C-1}$ all to vanish.

We note first that in view of (iii)

$$a_\nu(n+1) \sim a_\nu(n)$$

and by (vii) the $|a_\nu(n)|$ and $a(n)$ are bounded above and below. Thus (v) gives

$$n^{1-2p/C}a_{n+j} = \sum_{\nu=0}^{C-1}\left[\frac{a_\nu(n)}{\Gamma(2p/C)}\, e^{-i(n+j)\theta_\nu}\right]+o(1), \tag{5.4}$$

as $n\to\infty$ for $j = 0$ to $C-1$.

Suppose now that $a_{n+j} = 0$ for $j = 0$ to $C-1$, and some arbitrarily large n. We multiply (5.4) by $e^{i(n+j)\theta_0}$ for each j, $0 \leqslant j \leqslant C-1$ and add. This gives

$$\sum_{\nu=0}^{C-1}\frac{a_\nu(n)}{\Gamma(2p/C)}\left[\sum_{j=0}^{C-1} e^{-i(n+j)(\theta_\nu-\theta_0)}\right] = o(1).$$

Also in view of (vi)

$$\sum_{j=0}^{C-1} e^{-i(n+j)(\theta_\nu-\theta_0)} = \begin{cases} C, & \nu = 0, \\ 0, & \nu = 1 \text{ to } C-1. \end{cases}$$

Thus we deduce that

$$a_0(n) = o(1),$$

for some arbitrarily large n, which contradicts (vii). This shows that in (1.1) we must have $n_{\nu+1}-n_\nu \leqslant C$ and so $n_{\nu+1}-n_\nu = C$ for all large ν, so that $n_\nu = C\nu+B$ for large ν.

It follows that

$$f(z) = P(z)+\sum_{\nu=\nu_1}^{\infty} a_{n_\nu} z^{C\nu+B},$$

where $P(z)$ is a polynomial. If we set $\omega = \exp(2\pi i/C)$, this gives

$$f(\omega z) = P(\omega z) + \sum_{\nu=\nu_1}^{\infty} a_{n_\nu} \omega^{C\nu+B} z^{C\nu+B}, \tag{5.5}$$
$$= \omega^B f(z) + O(1),$$

as $|z| \to 1$ in any manner.

Suppose now that the numbers $r_\nu(R)$ are defined as in Lemma 10. It follows from Theorem 2 (vii) that $(1-r_\nu)(1-r_0)$ is bounded above and below as $R \to \infty$. Hence in view of Lemma 10 (iii) and (5.5) we have as $R \to \infty$

$$1 = \left| \frac{f(r_\nu e^{i\theta_\nu})}{f(r_0 e^{i\theta_0})} \right| \sim \left| \frac{f(r_\nu e^{i\theta_0})}{f(r_0 e^{i\theta_0})} \right| \sim \left(\frac{1-r_\nu}{1-r_0} \right)^{2p/C}.$$

Now Lemma 10 (ii) shows that for each fixed ν

$$R^{C^2}(1-r_\nu)^{2pC} \to \beta_3,$$

i.e.

$$|f(r_\nu e^{i\theta_\nu})| \sim \beta_3^{1/C^2}(1-r_\nu)^{-2p/C}$$

as $r_\nu \to 1$. This yields (vii') with $\beta = \beta_3^{1/C^2}$.

It remains to complete the proof of (viii). We note that by (5.5)

$$a_\nu(n) = \omega^{B\nu} a_0(n) + o(n^{(1-2p/C)}).$$

Thus Lemma 10 (v) gives

$$a_n = \frac{a_0(n) n^{2p/C-1} e^{-in\theta_0}}{\Gamma(2p/C)} \Big[\sum_{\nu=0}^{C-1} \omega^{(B-n)\nu} + o(1)\Big].$$

If $(B-n)$ is a multiple of C this gives

$$|a_n| \sim \frac{C|a_0(n)| n^{2p/C-1}}{\Gamma(2p/C)} \sim \frac{\beta C n^{2p/C-1}}{\Gamma(2p/C)}$$

in view of (vii'). This proves (viii) and completes the proof of Theorem 2.

6. It is reasonable to ask to what extent the condition $p > 4C$ is essential for (vii') and (viii). I am not able to answer this question as far as (vii') is concerned but (viii) is certainly false if $p < 4C$. We have in fact

THEOREM 3. *Suppose that $g(z) = (1-z)^{-1}$ and let b_n be an arbitrary sequence of complex numbers such that*

$$\sum_1^{\infty} |b_n| \leqslant 1, \quad \sum_1^{\infty} n|b_n|^2 < \frac{p}{C}$$

where $0 < p < C/4$. *Then the function*

$$f(z) = [g(z^C)]^{2p/C} + 4 + \sum_1^\infty b_n z^{Cn}$$

satisfies the hypotheses of Theorem 1 *and*

$$(1-r)^{2p/C} M(r, f) \to C^{-2p/C}, \quad \text{as} \quad r \to 1.$$

However we can choose the b_n *so that*

$$\overline{\lim_{\nu\to\infty}} (n_{\nu+1} - n_\nu) = \infty.$$

We suppose first that $C = 1$, and $p < \frac{1}{4}$, and set $w = G(z) = g(z)^{2p} + 4$. Then $w = g(z)$ maps $|z| < 1$ (1.1) conformally a subset of the halfplane $|\arg(w)| < \pi/2$, so that $G(z)$ maps $|z| < 1$ (1.1) conformally onto a subset of the sector

$$|\arg(w-4)| < p\pi$$

in the w plane. Let $w = 4 + te^{i\theta}$ be a point in this sector so that $|\theta| < p\pi < \pi/4$. Thus

$$|w|^2 = 16 + 8t\cos\theta + t^2 > (t+2)^2.$$

Thus $t < |w| - 2$, and the area $A(R)$ of the part of our sector in the disk $|w| < R$ is zero if $R < 2$, and is at most $p\pi(R-2)^2$ if $R > 2$.

Consider now a sequence b_n satisfying the hypotheses of Theorem 3 and set

$$\varphi(z) = \sum_1^\infty b_n z^n, \qquad f(z) = G(z) + \varphi(z).$$

Let $E(R)$ be the part of $|z| < 1$ in which $|f(z)| < R$. Clearly

$$|\varphi(z)| \leqslant \sum_1^\infty |b_n| \leqslant 1, \qquad |z| < 1,$$

so that in $E(R)$ we have

$$|G(z)| \leqslant |f(z)| + |\varphi(z)| < R + 1.$$

Thus

$$\iint_{E(R)} |G'(z)|^2 dxdy < A(R+1) \begin{cases} < p\pi(R-1)^2, & R > 1, \\ = 0, & R \leqslant 1. \end{cases}$$

Also

$$\iint_{E(R)} |\varphi'(z)|^2 dxdy \leqslant \iint_{|z|<1} |\varphi'(z)|^2 dxdy = \pi \sum_1^\infty n|b_n|^2 \leqslant p\pi.$$

Again if $R > 1$, we have

$$\iint\limits_{E(R)} |f'(z)|^2 dxdy \leqslant \iint\limits_{E(R)} \{|\varphi'(z)|^2 + |G'(z)|^2 + 2|G'(z)|\,|\varphi'(z)|\}\, dxdy,$$

and

$$\iint\limits_{E(R)} |G'(z)|\,|\varphi'(z)|\, dxdy \leqslant \Big\{ \iint\limits_{E(R)} |G'(z)|^2 dxdy\Big\}^{1/2} \Big\{\iint\limits_{E(R)} |\varphi'(z)|^2 dxdy\Big\}^{1/2}$$

$$\leqslant p\pi(R-1).$$

Thus if $R > 1$ we have finally

$$\iint\limits_{E(R)} |f'(z)|^2 dxdy \leqslant p\pi[1+2(R-1)+(R-1)^2]^2 = p\pi R^2.$$

This inequality remains valid for $R \leqslant 1$, since in this case $E(R)$ is void. In fact, we have for $|z| < 1$

$$|f(z)| \geqslant |G(z)| - |\varphi(z)| \geqslant 4 - 1 = 3.$$

Thus $f(z)$ is mean p-valent for $|z| < 1$.

We have

$$G(z) = (1-z)^{-2p} + 4 = \sum_0^\infty g_n z^n,$$

where

$$g_n = \frac{2p(2p+1)\ldots(2p+n-1)}{n!} \sim \frac{n^{2p-1}}{\Gamma(2p)}, \qquad \text{as} \qquad n \to \infty.$$

Since $p < \frac{1}{4}$, we can select a sequence $m = m_\nu$ of positive integers such that

$$\sum_{\nu=1}^\infty m_\nu g_{m_\nu}^2 < p, \qquad \sum_1^\infty g_{m_\nu} < 1,$$

and set $b_n = -g_n$, if $n = m_\nu$ for some ν and $b_n = 0$ otherwise. We ask in addition that for any integer d, we have $m_{\nu+d} = m_\nu + d$, infinitely often. Then

$$f(z) = \sum_0^\infty a_n z^n,$$

where $a_n = 0$ if $n = m_\nu$, and $a_n > 0$ otherwise. Also

$$M(r, f) \sim M[r, G(z)] \sim (1-r)^{-2p}.$$

This proves Theorem 3 if $C = 1$.

In the general case we write p/C instead of p and $f(z^C)$ instead of $f(z)$. For each root of the equation $f(z) = w$ in $|z| < 1$ there are precisely C roots of the equation $f(z^C) = 1$ in $|z| < 1$, so that the resulting function is still mean p-valent. Also

$$M[r, f(z^C)] = M(r^C, f(z)) \sim (1-r^C)^{-2p/C} \sim C^{-2p/C}(1-r)^{-2p/C},$$

as required. Finally,

$$f(z^C) = \sum_0^\infty a_n z^{nC}$$

so that the indices of non-zero terms are all multiples of C. If n_ν are the successive indices of non-zero terms we deduce that $n_{\nu+1} - n_\nu = C$, and since $a_{n+1} = \ldots = a_{n+d} = 0$, for infinitely many n we must have $n_{\nu+1} - n_\nu \geqslant dC$ for infinitely many n and any fixed d. This completes the proof of Theorem 3.

REFERENCES

[1] B. Eke, *Ph. D. Thesis,* London 1965 (to be published in Journal d'Analyse Mathématique).

[2] W. K. Hayman, *A characterisation of the maximum modulus of functions regular at the origin,* Journal d'Analyse Mathématique 1 (1951), p. 155-179.

[3] — *Multivalent functions,* Cambridge 1958.

[4] A. E. Ingham, *Some trigonometrical inequalities with applications to the theory of series,* Mathematische Zeitschrift 41 (1936), p. 367-379.

[5] J. E. Littlewood, *On the coefficients of schlicht functions,* Quarterly Journal of Mathematics (Oxford) (2) 9 (1938), p. 14-20.

[6] Ch. Pommerenke, *Über die Mittelwerte und Koeffizienten multivalenter Funktionen,* Mathematische Annalen 145 (1961/62), p. 285-296.

[7] — *On the coefficients and Hankel determinants of univalent functions,* Journal of the London Mathematical Society 40 (1966).

[8] D. C. Spencer, *On finitely mean valent functions,* Proceedings of the London Mathematical Society (2) 47 (1941), p. 201-211.

MPERIAL COLLEGE, LONDON

Reçu par la Rédaction le 3. 1. 1966