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POINTWISE CONVERGENCE OF MULTIPLIER OPERATORS

BY

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Introduction. Let m be a bounded function on $\mathbf{R}^n$ and define the multiplier operator T by the Fourier transform equation $(Tf)^\wedge(\xi) = m(\xi)\widehat{f}(\xi)$. Such an operator is always bounded on L^2, but m must satisfy some stronger condition in order that T be bounded on other spaces. Let $[x]$ be the greatest integer function. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, let $|\alpha| = \alpha_1 + \ldots + \alpha_n$. We say a multiplier satisfies the condition $M(s)$, $1 < s < \infty$, if

$$\sup_{|\alpha|\le[n/s]+1} \sup_{R>0} \int_{\{R<|\xi|<2R\}} |D^\alpha m(\xi)|^s \, d\xi < \infty .$$

The condition $M(2)$ was introduced by Hörmander [3] in 1960. He proved that such a condition guarantees T is bounded on all the L^p spaces, $1 < p < \infty$. (See [1, 5, 7] for $s \neq 2$.) We call a multiplier satisfying an $M(s)$ condition a *Hörmander multiplier.*

There are two standard methods for proving the boundedness of these multiplier operators. Hörmander's approach is to consider a sequence of bounded operators which converge to T and define T as the limit of such operators; the other is to use Littlewood–Paley theory (see [6]). In either case, given a function f in L^p, Tf is defined as an L^p function.

Hörmander's construction involves a sequence of multipliers m_N defined by multiplying m by smooth cutoff functions. Let T_N be the multiplier operator associated to m_N. Hörmander proved that the operators $\{T_N\}_{N=1}^{\infty}$ are bounded on L^p with norms independent of N. It follows from this that $\|Tf\|_p \le C\|f\|_p$ for all $f \in S$ and hence, by continuity, T defines a bounded operator on L^p. In analogy with the study of singular integral operators, we consider the pointwise convergence of the operators T_N to T for $f \in L^p$. To prove the pointwise convergence, we use the maximal multiplier operator T^*, where $T^*f = \sup_N |T_N f|$. The main result of this paper is the following theorem.

THEOREM 1. *If $m \in M(s)$, $1 < s \le 2$, then there is a constant $C = C(p)$ so that*

$$\|T^* f\|_p \le C(p)\|f\|_p , \quad 1 < p < \infty ,$$

and

$$|\{x : |T^*f(x)| > \lambda\}| \le \frac{C(1)}{\lambda}\|f\|_{H^1}. \ \blacksquare$$

As a consequence of this result and standard arguments, we get the next theorem.

THEOREM 2. *If* $m \in M(s)$, $1 < s \le 2$, *then* $\mathcal{F}^{-1}(m_N\widehat{f})$ *converges to* $\mathcal{F}^{-1}(m\widehat{f})$ *almost everywhere for all* $f \in L^p$, $1 < p < \infty$, *and* $f \in H^1$. ■

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Results. We begin by recalling Hörmander's construction. Let $\Phi \in C^\infty$ be nonnegative, with $\operatorname{supp}(\Phi) \subset \{1/2 < |\xi| < 2\}$ and $\{1/\sqrt{2} < |\xi| < \sqrt{2}\} \subset \{\Phi(\xi) > 0\}$. Set

$$\varphi(\xi) = \Phi(\xi) / \sum_{j=-\infty}^{\infty} \Phi(2^{-j}\xi)$$

so that, for $\xi \ne 0$,

$$\sum_{j=-\infty}^{\infty} \varphi(2^{-j}\xi) \equiv 1.$$

Define φ_N by $\varphi_N(\xi) = \sum_{j=-N}^{N} \varphi(2^{-j}\xi)$, for $N = 1, 2, \dots$ Notice that $0 \le \varphi_N \le 1$, $\operatorname{supp}(\varphi_N) \subset \{2^{-N-1} < |\xi| < 2^{N+1}\}$, and $\{2^{-N} \le |\xi| \le 2^N\} \subset \{\varphi_N(\xi) = 1\}$.

Define a sequence of multipliers $\{m_N\}_{N=1}^{\infty}$ by $m_N = \varphi_N m$ and let T_N be the multiplier operator associated to m_N. Note that for $f \in \mathcal{S}$,

$$\|T_N f - Tf\|_\infty \le \|(m_N - m)\widehat{f}\|_1 \to 0$$

since $m_N \to m$ pointwise and $\|m_N - m\|_\infty \le 2\|m\|_\infty$. Thus, $T_N f(x) \to Tf(x)$ almost everywhere.

Since we are interested in the pointwise limit of $T_N f$, the first question to consider is whether $T_N f$ is a well defined function for almost every x. Suppose that m is a Hörmander multiplier and $f \in L^p$. Then $Tf \in L^p$ so that $(Tf)^\wedge$ is a temperate distribution. Since $\varphi_N \in C_c^\infty$, it follows that the inverse Fourier transform of $\varphi_N (Tf)^\wedge$ is a C^∞ function [4, p. 191]. Since

$$(1) \qquad (T_N f)^\wedge = m_N \widehat{f} = (\varphi_N m)\widehat{f} = \varphi_N(m\widehat{f}) = \varphi_N (Tf)^\wedge,$$

we see that $T_N f \in C^\infty$. Further, this shows that in order to study the pointwise convergence of $T_N f$ it is enough to consider summability defined in terms of the functions φ_N.

Therefore, we consider the operators S_N defined by $(S_N f)^\wedge = \varphi_N \widehat{f}$ and let S^* be the associated maximal operator, $S^* f = \sup_N |S_N f|$. Our results about multiplier operators are a consequence of the theorem below.

THEOREM 3. *There is a constant $C = C(p)$ so that*

$$\|S^*f\|_p \le C(p)\|f\|_p, \quad \text{for } 1 < p < \infty,$$

and

$$|\{x \in \mathbf{R}^n : S^*f(x) > \lambda\}| \le \frac{C(1)}{\lambda}\|f\|_1 . \blacksquare$$

As above, $S_N f \to f$ almost everywhere for all $f \in \mathcal{S}$. Thus we get the following result.

COROLLARY 4. *If $f \in L^p$, $1 \le p < \infty$, then $S_N f$ converges to f almost everywhere.* ■

Proof (of Theorem 3). The function φ_1 is identically 1 on $\{1/2 \le |\xi| \le 2\}$ and 0 outside of $\{1/4 \le |\xi| \le 4\}$. Define two smooth functions, κ and η, by

$$\kappa(\xi) = \begin{cases} 1 - \varphi_1(\xi) & \text{for } |\xi| \le 1, \\ 0 & \text{for } |\xi| \ge 1, \end{cases} \qquad \eta(\xi) = \begin{cases} 1 & \text{for } |\xi| \le 1, \\ \varphi_1(\xi) & \text{for } |\xi| \ge 1. \end{cases}$$

Note that $\varphi_1(\xi) = \eta(\xi) - \kappa(\xi)$; moreover, $\varphi_N(\xi) = \eta(\xi/2^{N-1}) - \kappa(2^{N-1}\xi)$.

Fix $\varepsilon > 0$ and define the operator $\mathcal{H}_\varepsilon$ by $(\mathcal{H}_\varepsilon f)^\wedge(\xi) = \eta(\varepsilon\xi)\widehat{f}(\xi)$ for $f \in \mathcal{S}$. Let $H = \mathcal{F}^{-1}(\eta)$ and set $H_\varepsilon(y) = \varepsilon^{-n}H(y/\varepsilon)$. Then $\mathcal{H}_\varepsilon f(x) = (H_\varepsilon * f)(x)$. Since $\eta \in \mathcal{S}$, it follows that H and the least decreasing radial majorant of H are both in L^1. Thus, the operator $\mathcal{H}^*$ defined by $\mathcal{H}^*f(x) = \sup_{\varepsilon>0}|\mathcal{H}_\varepsilon f(x)|$ satisfies the estimate $\mathcal{H}^*f(x) \le C \cdot Mf(x)$, where Mf is the Hardy–Littlewood maximal function [6, p. 62].

Next, consider the operator $\mathcal{K}_\varepsilon$ defined by $(\mathcal{K}_\varepsilon f)^\wedge(\xi) = \kappa(\xi/\varepsilon)\widehat{f}(\xi)$ for $f \in \mathcal{S}$. Let $K = \mathcal{F}^{-1}(\kappa)$. Thus, $\mathcal{K}_\varepsilon f(x) = (K_{1/\varepsilon} * f)(x)$ and as above we see that $\sup_{\varepsilon>0}|\mathcal{K}_\varepsilon f(x)| \le C \cdot Mf(x)$.

Observe that for $f \in \mathcal{S}$,

$$\begin{aligned} S_N f(x) &= (\mathcal{F}^{-1}(\varphi_N \widehat{f}))(x) = (\mathcal{F}^{-1}(\{\eta(\xi/2^{N-1}) - \kappa(2^{N-1}\xi)\}\widehat{f}))(x) \\ &= (\mathcal{H}_{2^{1-N}} f)(x) - (\mathcal{K}_{2^{1-N}} f)(x) . \end{aligned}$$

This implies that $|S_N f(x)| \le C \cdot Mf(x)$. It follows that S^* is a bounded operator on L^p, for $p > 1$, and weak-type (1, 1). This completes the proof of Theorem 3. ■

The proof of Theorem 1 is an easy consequence of Theorem 3 and inequality (1). Let $f \in \mathcal{S}$. By (1) we have $T_N f = S_N(Tf)$, so that $T^*f = S^*(Tf)$. This implies that $\|T^*f\|_p \le C\|f\|_p$, for $p > 1$. If $f \in H^1$, then $Tf \in L^1$ (in fact, it is in H^1), so that T^*f is in weak-L^1. ■

The proof of Theorem 3 shows that $S^*f(x) \le C \cdot Mf(x)$ for almost every x. It follows that the conclusion of the theorem is true on weighted L^p spaces when the weight w satisfies the A_p condition. We define the space L^p_w as the collection of measurable functions f so that $\|f\|_{p,w} = (\int |f(x)|^p w(x)\,dx)^{1/p} < \infty$.

COROLLARY 5. *Let* $1 < p < \infty$ *and* $w \in A_p$. *There is a constant* $C(p,w)$ *so that*

$$\|S^* f\|_{p,w} \le C(p,w)\|f\|_{p,w}, \quad \textit{for } 1 < p < \infty,$$

and

$$w(\{x \in \mathbf{R}^n : S^* f(x) > \lambda\}) \le \frac{C(1,w)}{\lambda}\|f\|_{1,w}. \ \blacksquare$$

Since these pointwise convergence results depend only on the L^p boundedness of the multiplier operator and properties of the Hardy–Littlewood maximal function, analogs of Theorems 1 and 2 are also true for functions in weighted L^p spaces, whenever the weight satisfies the appropriate conditions for the multiplier and the maximal function. We refer the interested reader to [5, 7]. Note that the conditions on the weight used for the boundedness of the multiplier operators on L^p_w guarantee the boundedness of the maximal function.

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