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# THE EQUIVALENCE OF SOME BANACH SPACE PROBLEMS

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**0. Introduction.** The problem of whether every Banach space (always infinite dimensional in this article) has a separable quotient appears to have been considered since 1932 although we cannot find an explicit mention of it earlier than 1969 [10]. In [9] and [10], it is proved that $C(X)$ has a separable quotient, where $X$ is compact Hausdorff.

In Section 1 of this article we give six other properties equivalent to this one. Most of these are known, some can be found in such articles as [10] and [2]. The main result, which we believe to be new, is that a Banach space has a separable quotient if and only if it has a dense non-barrelled subspace. (Thus it may be significant that there exist spaces of all of whose dense subspaces are barrelled [4].) We also give some simplified proofs of the known results. In order to make the work self-contained we prove all results.

**1.** We now list 7 properties which a Banach space $E$ may have. Later we prove them to be equivalent. It is unknown whether every Banach space has these properties. (**P 1016**)

$P_1$. *$E$ has a separable quotient.*

That is, $E$ can be mapped onto a separable Banach space.

$P_2$. *$E$ has a dense non-barrelled subspace.*

Dense barrelled subspaces are plentiful, however (see [13]).

An *$S_\sigma$-subspace* is the union of a strictly increasing sequence of closed subspaces. An *$S_\sigma$-space* is a locally convex space which is an $S_\sigma$-subspace of itself.

$P_3$. *$E$ has a dense $S_\sigma$-subspace.*

A *BE-subspace* is a proper subspace of $E$ which can be given a larger complete norm, i.e., it is a Banach space properly and continuously included in $E$. For example, $l$ is a $Bc_0$-subspace of $c_0$.

$\mathrm{P}_4$. *$E$ has a dense BE-subspace.*

If in the preceding definition Banach is replaced by Fréchet, we have an *FE-subspace.*

$\mathrm{P}_5$. *$E$ has a dense locally convex FE-subspace.*

$\mathrm{P}_6$. *There exists a sequence $\{f_n\}$ in $E'$ with $\|f_n\| = 1$ for all $n$ and $\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} f_i^{\perp}$ dense in $E$.*

Note that $f_n \to 0$ weak-*. Let us define a *normal sequence* to be a sequence $\{f_n\} \subset E'$ with $\|f_n\| = 1, f_n \to 0$ weak-*. It has recently been proved independently by A. Nissenzweig and B. Josefson (see [6]) that every Banach space has a normal sequence. We may express $\mathrm{P}_6$ as follows:

*There exists a normal sequence $\{f_n\}$ with $\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} f_i^{\perp}$ dense.*

$\mathrm{P}_7$. *There exist closed subspaces $A$ and $B$ of $E$ with $A+B$ a dense proper subset of $E$.*

Remark on $\mathrm{P}_7$. It follows from $\mathrm{P}_7$ that *$A$ and $B$ can be found so that $A \cap B = \{0\}$, i.e., $A$ and $B$ are quasicomplementary.* To see this, observe that $\mathrm{P}_7$ implies $\mathrm{P}_1$ (see below), thus, as remarked in [10], p. 188 (2), $E$ has a quasicomplemented separable subspace. From this it also follows that *$E$ has a quasicomplemented separable subspace if and only if it has a quasicomplemented subspace.*

The equivalence of $\mathrm{P}_2$, $\mathrm{P}_4$ and $\mathrm{P}_5$ was given in [2], p. 512-513, and that of $\mathrm{P}_1$ and $\mathrm{P}_3$ in [5], p. 85. Our proofs are somewhat simpler. That $\mathrm{P}_3$ implies $\mathrm{P}_2$ follows from [1], p. 274; the special case needed here is given an easy proof.

**1.1.** $\mathrm{P}_4$ *implies* $\mathrm{P}_5$; $\mathrm{P}_5$ *implies* $\mathrm{P}_2$.

Let $F$ be a dense locally convex $FE$-space. If $(F, T_E)$ were barrelled, Ptak's open mapping theorem applied to the identity map from $(F, T_F)$ to $(F, T_E)$ would make $T_E = T_F$, and so $F$ would be a closed subspace of $E$, hence $F = E$.

Note. This argument fails if we omit "locally convex" from $\mathrm{P}_5$; indeed, $E$ may have a barrelled subspace which is a dense $FE$-subspace.

**1.2.** $\mathrm{P}_2$ *implies* $\mathrm{P}_4$.

Let $S$ be a dense non-barrelled subspace of $E$. Let $B$ be a bounded barrel in $S$ which is not a neighborhood of 0. (If $B$ is not bounded, replace it by $B \cap D$, $D$ being the unit disc.) Let $F$ be the span of $\bar{B}$, the closure of $B$ in $E$. Then $\bar{B}$ is not a neighborhood of 0 in $F$ $(B = \bar{B} \cap S)$. Now $\bar{B}$ is an absolutely convex, bounded, closed (hence complete) subset of $E$, and so its gauge $p$ is a complete norm for $F$ which is larger than the relative topology of $E$ (see [16], 6-1-17). Thus $F$ is a dense $BE$-subspace of $E$. ($F \neq E$, since $\bar{B}$ not a neighborhood of 0 in $F$ implies $F$ is not barrelled.)

Note. W. H. Ruckle suggests the following proof of 1.2. Let $\{f_n\}$ be unbounded, but convergent on a dense subspace. Then

$$F = \{x\colon \lim f_n(x) \text{ exists}\}$$

is a $BE$-subspace with $\|x\|_1 = \|x\| + \sup |f_n(x)|$.

**1.3.** $P_1$ *implies* $P_3$.

This is clear since $P_3$ is true for the separable quotient.

**1.4.** $P_3$ *is equivalent to* $P_6$.

If $\bigcup S_n$ is dense, let $\|f_n\| = 1, f_n = 0$ on $S_n$.

**1.5.** THEOREM. *Let $E$ be a quasibarrelled topological vector space. The following are equivalent:*

(i) *$E$ is barrelled.*

(ii) *$E$ is $\omega$-barrelled (every $w^*$ bounded sequence is equicontinuous).*

(iii) *$E$ is c-barrelled (every $w^*$ Cauchy sequence is equicontinuous).*

(iv) *$E$ is sequentially barrelled (every $w^*$ convergent sequence is equicontinuous).*

(v) *$(E', w^*)$ is sequentially complete.*

This improves the result of De Wilde [3], Theorem 2.7, that (v) implies (i) for metrizable spaces, since metrizable implies quasibarrelled. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial. The implication (iv) $\Rightarrow$ (i) is proved by Webb [14], Proposition 4.1. The implication (iii) $\Rightarrow$ (v) is obvious. Finally, assuming (v), let $A \subset E'$ be $w^*$-bounded. Then $A$ is $\beta(E', E)$-bounded by the Banach-Mackey theorem (see [8], 20.11(8)), hence it is equicontinuous, since $E$ is quasibarrelled. Thus $E$ is barrelled.

We note that (iii) $\equiv$ (v) under the weaker hypothesis that $E$ is relatively strong. This follows from [7], Theorems 2.4 and 3.1.

**1.6.** LEMMA. *Let $X$ be a metrizable $S_\sigma$-space. Then $X'$ cannot be $w^*$-sequentially complete.*

Let $X = \bigcup S_n$, $x^n \in S_n \setminus S_{n-1}$, and $x^n \to 0$. Let $f_n \in X'$ with $f_1 = 0$, $f_n = f_{n-1}$ on $S_{n-1}$, and $f_n(x^n) = 1$ for $n = 2, 3, \ldots$ For all $x$, $\{f_n(x)\}$ is eventually constant but its pointwise limit $f$ is not continuous, since $f(x^n) = 1$ for all $n$.

**1.7.** COROLLARY. *A metrizable $S_\sigma$-space cannot be barrelled.*

This generalizes the known result that a metrizable space of countable dimension cannot be barrelled. More precisely, one can prove that a barrelled $S_\sigma$-space must have the inductive limit topology by the closed subspaces and inclusion maps.

**1.8.** $P_3$ *implies* $P_2$.

This follows from 1.7.

**1.9.** THEOREM. *Let $E$ have a dense non-barrelled subspace; then $E$ has an increasing sequence $\{S_n\}$ of closed subspaces with $\bigcup S_n$ dense and* $\dim S_n/S_{n-1} = 1$ *for each $n$.*

Let $F$ be, as in 1.2, a dense subspace having a barrel $B$ which is not a neighborhood of 0 in $F$ and which is closed in $E$. Let

$$B_1 = B, \quad B_n = B+\Big\{\sum_{i=1}^{n-1} a_i x^i\colon |a_i| \leqslant 1\Big\},$$

where $x^1, x^2, \ldots$ are about to be defined. Each $B_n$ is closed and not absorbing. (Otherwise, it would be a barrel in $E$, hence a neighborhood of 0, and so each $B_i$ $(i< n)$ would be a neighborhood of 0 in its span; but $B_1$ is not.) We first choose $x^1 \notin 2B_1$, $\|x^1\| = 1$, and $f_1 \in E'$ with $f_1(x^1) = 1$, $|f_1| \leqslant 1/2$ on $B_1$. Since the span of $B_2$ has infinite codimension, it does not include $f_1^\perp$, thus there exists $x^2$ with $\|x^2\| = 1$, $f_1(x^2) = 0$, $x^2 \notin 4B_2$. Then $f_2 \in E'$ with $f_2(x^2) = 1$, $|f_2| \leqslant 1/4$ on $B_2$. The span of $B_3$ does not include $f_1^\perp \cap f_2^\perp$, thus there exists $x^3$ with $\|x^3\| = 1, f_1(x^3) = f_2(x^3) = 0$, $x^3 \notin 8B_3$, and $f_3 \in E'$ with $f_3(x^3) = 1$, $|f_3| \leqslant 1/8$ on $B_3$. In general,

$$\|x^n\| = 1, \quad f_i(x^n) = 0 \text{ for } i<n, \quad f_n(x^n) = 1, \quad |f_i| \leqslant \frac{1}{2^i}$$

on $B_i$, and so $|f_i(x^n)| \leqslant 1/2^i$ for $i > n$.

Now let $x \in B_1$. Fix a positive integer $k$. Let $a_1 = -f_{k+1}(x)$. So $|a_1| < 1/2^{k+1}$. Let

$$a_n = -f_{k+n}\Big(x+\sum_{i=1}^{n-1} a_i x^{k+i}\Big).$$

So $|a_n| \leqslant 1/2^{k+n-1}$ (by induction).

Let

$$z = \sum_{i=1}^{\infty} a_i x^{k+i};$$

then $\|z\| \leqslant 1/2^k$ and $f_{k+n}(x+z) = 0$ for all $n = 1, 2, \ldots$

Indeed,

$$f_{k+n}(x+z) = f_{k+n}\Big(x+\sum_{i=1}^{n-1} a_i x^{k+i}\Big)+a_n+\sum_{i=n+1}^{\infty} a_i f_{k+n}(x^{k+i}) = -a_n+a_n+0.$$

Thus, setting

$$S_k = \bigcap_{i=k+1}^{\infty} f_k^\perp$$

we have proved that $d(x, S_k) \leqslant 1/2^k$. Setting $S = \bigcup S_k$ we have $d(x, S) = 0$. Thus $S$ is dense in $B_1$, hence dense in its span, hence dense in $E$.

**1.10.** $P_2$ *implies* $P_1$.

With $\{S_n\}$ as in 1.9, let $S_n = S_{n-1} \oplus y^n$. Let $q\colon E \to F = E/S_1$ be the quotient map and suppose that, for some $v \in F'$, $v(qy^n) = 0$ for all $n$. Then, with $q'\colon F' \to E'$, $q'v$ vanishes on $\bigcup S_n$.

Indeed, for $x \in S_n$,

$$x = s^1 + \sum_{i=1}^{n-1} a_i y^i, \qquad q'v(x) = v(qx) = \sum_{i=1}^{n-1} a_i v(qy^i) = 0.$$

So we have $q'v = 0$. Since $q'$ is one-to-one, $v = 0$. Thus $\{qy^n\}$ is a total sequence and $F$ is separable.

**1.11.** $P_1$ *implies* $P_7$.

See the Remark following $P_7$.

**1.12.** $P_7$ *implies* $P_4$.

We make $A+B$ into a $BE$-space by means of the inductive topology by the inclusion maps, i.e.,

$$\|x\| = \inf\{\|a\| + \|b\|\colon\ x = a+b,\ a \in A,\ b \in B\}$$

(see [16], Section 13-4).

Thus the seven properties are equivalent.

**2.** We present some comments on the seven equivalent conditions. As noted above,

$(*)$ *every Banach space has a normal sequence.*

Previously, it had been known that each property $P_i$ implies $(*)$. In fact, for $E$ separable the unit disc of $E'$ is $w^*$-metrizable, thus $P_1$ implies $(*)$. If $F$ is a dense non-barrelled subspace, by 1.5 we can find an unbounded sequence $B \subset E'$ with $B$ pointwise bounded on $F$. Thus $P_2$ implies $(*)$. If $F$ is a dense $BE$-subspace and $i\colon F \to E$ is inclusion, $i'$ is not an isomorphism, thus $P_4$ implies $(*)$.

It is also easy to see that a reflexive space has a normal sequence, for if not, its unit disc, being weakly compact, is weakly sequentially compact (see, e.g., [15], 13.4, Example 2), and so the unit circumference would be norm compact.

**2.1.** Example. The implication $P_6 \Rightarrow (*)$ is trivial and leads to the hope that $(*)$ implies $P_6$, thus solving our main problem. This is false, since, e.g.,

$$E = c_0 \cap bs, \qquad \text{where } bs = \left\{x\colon\ \|x\| = \sup\left|\sum_{k=1}^{n} x_k\right| < \infty\right\}.$$

Here $\{f_n\}$, with $f_n(x) = x_n$, is (essentially) a normal sequence, but the set given in $P_6$ is not dense, since $E$ is not separable.

However, $P_6$ does show that the main problem is equivalent to the existence of a normal sequence with an additional property.

**2.2.** Remark. Another way in which (*) would solve the main problem is that it might be possible to find a sequence $\{a_n\}$ of real numbers, tending to infinity, such that $S_a = \{x\colon a_n f_n(x) \to 0\}$ is dense. Since $S_a$ cannot be barrelled, $P_2$ would hold. This also is not possible in general.

**2.3.** Example (A. K. Snyder). Let $E$ and $f_n$ be as in 2.1. Let $\{a_n\}$ be a strictly increasing sequence of positive numbers tending to infinity. We may assume that $a_1 \geqslant 1$. Let $\{p_n\}$ be a sequence of positive integers increasing to infinity with $p_n \leqslant a_n$ for all $n$. Let

$$q_n = 1 + \sum_{k=1}^{n-1} p_k.$$

Define $z \in E$ by $z_1 = 0$, and

$$z_i = \begin{cases} 1/p_{2k-1} & \text{for } q_{2k-1} < i \leqslant q_{2k}, \\ -1/p_{2k} & \text{for } q_{2k} < i \leqslant q_{2k+1}. \end{cases}$$

We assert that $d(z, S_a) \geqslant 1/4$ (see 2.2 for $S_a$). If this is false, let $\|x - z\| < 1/4$, $x \in S_a$. Now

$$\sum_{i=1}^{q_j} z_i = \begin{cases} 0 & \text{for } j \text{ odd}, \\ 1 & \text{for } j \text{ even}. \end{cases}$$

Thus

$$\left|\sum_{i=1}^{q_j} x_i\right| \begin{cases} < 1/4 & \text{for } j \text{ odd}, \\ > 3/4 & \text{for } j \text{ even}, \end{cases}$$

and so

$$\left|\sum \{x_i\colon q_{k-1} < i \leqslant q_k\}\right| > \frac{1}{2} \quad \text{for all } k. \tag{1}$$

Since $x \in S_a$, $p_n x_n \to 0$, thus, for large $k$, $q_{k-1} < i \leqslant q_k$ implies $|p_i x_i| < 1/2$. It follows that

$$\frac{1}{2|x_i|} > p_i \geqslant p_{q_{k-1}} \geqslant p_{k-1},$$

and so

$$\left|\sum \{x_i\colon q_{k-1} < i \leqslant q_k\}\right| \leqslant \frac{q_k - q_{k-1}}{2p_{k-1}} = \frac{1}{2},$$

contradicting (1).

**2.4.** Example (A. K. Snyder). However, the space of Example 2.3 does have a separable quotient for $q\colon E \to bs$ given by

$$qx = \left\{\sum_{k=n^2}^{(n+1)^2-1} x_k\right\}$$

is onto, and $bs$ is equivalent to $l^\infty$ which is known to have a separable quotient.

**2.5.** Remark. It can be checked that the space of Example 2.3 has a dense non-barrelled subspace independently of 2.3 and the fact that $P_1$ implies $P_2$; namely,

$$\{x\colon x_{2n} = 0 \text{ for almost all } n\}$$

is a dense $S_\sigma$-subspace. Moreover, the quotient of $c_0 \cap bs$ by $\{x\colon x_{2n} = 0$ for all $n\}$ is separable.

**2.6.** Remark. *The statement* $P_1$ *is equivalent to*

$P_8$. *For every set $S$ and closed subspace $A$ of $l_1(S)$ of infinite dimension and codimension there exists a closed subspace $B$ such that $A+B$ is a dense proper subspace.*

Let $E$ be a Banach space; then there exist $S$ and a subspace $A$ of $l_1(S)$ such that $E = l_1(S)/A$. If $A$ is finite dimensional, then $l_1(S) = E \oplus A$ and the map of $l_1(S)$ onto a separable space does the same for $E$. If $A$ is infinite dimensional, let $A+B$ be dense and proper. Then $Q[B]$ is a dense proper $BE$-subspace. Conversely, $P_8$ follows by applying $P_7$ to $l_1(S)/A$.

**3.** In this section we prove that every Banach space has a strictly larger barrelled norm and a larger non-barrelled norm.

**3.1.** Lemma. *Let $E$ be a barrelled metrizable space and $H$ a Hamel basis for $E$. Then all but a finite number of the coefficient functionals are discontinuous.*

Suppose that there is a sequence $\{P_n\}$ of continuous coefficient functionals corresponding to $\{h^n\} \subset H$. For an arbitrary sequence $\{s_n\}$ of positive real numbers, $\{s_n P_n\}$ is $w^*$-null, hence equicontinuous. Choose scalars $a_n \neq 0$ with $a_n h^n \to 0$. The set $\{a_n h^n\}$ is bounded, hence $\{s_n P_n(a_n h^n)\} = \{s_n a_n\}$ is a bounded sequence for any choice of $\{s_n\}$. This is impossible.

**3.2.** Theorem. *Every Banach space $E$ has a larger non-barrelled norm and a strictly larger barrelled norm.*

Let $H$ be a Hamel basis for $E$ and put

$$p(x) = \sum |t|\cdot\|h\| \quad \text{for } x = \sum th,\ h \in H.$$

For each $h \in H$ the corresponding coefficient functional $P$ satisfies $|P(x)| \leqslant p(x)$ and it follows from 3.1 that $(E, p)$ is not barrelled. Next, let $f$ be an unbounded linear functional on $E$ and write $q(x) = \|x\| + |f(x)|$. Then $(E, q)$ is the direct sum of a one-dimensional space and $F = (f^\perp, q)$, hence it is barrelled, since, as we now show, $F$ is barrelled. Indeed, $q(x) = \|x\|$ for $x \in F$, and so $F$ is a maximal subspace of the Banach space $E$, hence it is barrelled (see [13]).

The second result of 3.2 says that certain barrelled spaces have smaller complete norms. Others do not, for example, a space of countable dimension or a space such as $(l^{1/2}, \|\cdot\|_1)$ which has a larger Fréchet topology. $(\|x\|_1 = \sum |x_n|.)$

**3.3.** *Every finite-codimensional subspace $S$ of a Banach space $E$ is a barrelled space which has a smaller complete norm.*

We may assume that $S$ is of the form $\bigcap \{f_i^\perp : i = 1, 2, \dots, m\}$, where each $f_i$ is an unbounded linear functional on $E$,

$$E = S \oplus \{x^1, x^2, \dots, x^m\} \qquad \text{and} \qquad f_i(x^j) = \delta_i^j.$$

For $a \in E$ and $f \in E'$, write $(a \otimes f)(x) = f(x)a$. Now set

$$P = I - \sum_{i=1}^{m} x^i \otimes f_i,$$

a projection of $E$ onto $S$. Let $g_1, g_2, \dots, g_m \in E'$ with $g_i(x^j) = \delta_i^j$, $H = \bigcap g_i^\perp$. Then $A = P|H$ is an algebra isomorphism of $H$ onto $S$ with the inverse

$$B = Q|S, \qquad \text{where } Q = I - \sum x^i \otimes g_i.$$

Since $B$ is continuous, it transfers from $H$ to $S$ a smaller norm $\|\cdot\|_B$ such that $(S, \|\cdot\|_B)$ is equivalent to $H$ (the original norm of $E$ on $H$).

**3.4.** Example. Take $m = 1$ in the construction of 3.3 and omit the subscripts and superscripts. Assume that for $y \in E$, $x = h + ax$, and $h \in H$ we have $\|x\| = \|h\| + |a|$. (This is equivalent to the original norm.) Then, for $y \in H$,

$$\|y\|_A = \|Ay\| = \|y - f(y)x\| = \|y\| + |f(y)|.$$

This is precisely the norm defined in the second part of 3.2 and gives another proof that it is barrelled. In fact, $(H, \|\cdot\|_A)$ is equivalent to $S$ (original norm of $E$ on $S$).

**3.5.** COROLLARY. *The second norm $q$ defined in* 3.2 *has the property that $(E, q)$ is of codimension* 1 *in its completion.*

In 3.4, $(H, \|\cdot\|_A)$ is equivalent to $S$ which is a dense maximal subspace.

**3.6.** Example. *A vector space $E$ with two non-comparable complete norms $p$ and $q$ such that $(E, p + q)$ is barrelled.*

Let $(E, p)$ be a Banach space, $S$ a dense maximal subspace, and $q$ any smaller complete norm for $S$ (e.g., 3.3). We extend $q$ to $E$ by $q(y + ax) = q(y) + |a|$ for $y \in S$, and fixed $x \notin S$. Then $p$ and $q$ are complete and non-comparable ($S$ is $q$-closed in $E$). To see that $(E, p + q)$ is barrelled note that $p$ and $p + q$ are equivalent on $S$, since $q \leqslant p$ on $S$. Thus $(S, p + q)$

is barrelled. Now $S$ is a closed subspace of $(E, q)$, hence of $(E, p+q)$, and so $(E, p+q)$ is the direct sum of a barrelled subspace and a one-dimensional one.

**4.** The following result is a step in the direction of $P_2$.

**4.1.** THEOREM. *Every Banach space $E$ has a dense non-Baire subspace $S$; indeed, $S$ may be included in the union of a sequence of closed proper subspaces of $E$.*

Let $\{x^i\} \subset E$ and $\{f_i\} \subset E'$ be bi-orthogonal. For each $x \in E$, $\varepsilon > 0$, and $i = 1, 2, 3, \ldots$, choose one scalar $a_i$ such that $a_i + f_i(x)$ is rational and $|a_i| < \varepsilon/2^i$. Let

$$A = \{x^i\} \cup \left\{x + \sum a_i x^i\right\},$$

where $x \in E$, $\varepsilon > 0$, and the $a_i$ are chosen for $x, \varepsilon$ as specified. Now fix $n > 1$ and write

$$P = \sum_{i=1}^{n} x^i \otimes f_i$$

(as in 3.3), a projection onto $U = \operatorname{span}\{x^1, x^2, \ldots, x^n\}$. Let $C$ be a subset of $PA$ which has fewer than $n$ members. Then $H = P^{-1}$ $(\operatorname{span} C)$ is a closed subspace of $E$ and is proper. (Choose $y \in U \setminus \operatorname{span} C$. Then $Py = y \notin \operatorname{span} C$, so $y \notin H$.) Note that $PA$ is countable, and so there are at most countably many such subsets $C$, hence countably many $H$, and the union of all such $H$ contains each point $z \in \operatorname{span} A$ which is a linear combination of fewer than $n$ members of $A$. ($Pz \in C$ for some $C$, so $z \in P^{-1}C$.) Since every member of $\operatorname{span} A$ is of this type for some $n$, we thus have $\operatorname{span} A$ covered by the union of countably many proper closed subspaces. Finally, $\operatorname{span} A$ is obviously dense.

**4.2.** Remark. With the notation of 4.1, let

$$B = \{x \in E \colon f_i(x) \text{ is rational for } i = 1, 2, \ldots\}.$$

It is shown in [12] that the span of $B$ is also dense and non-Baire and it is barrelled as well.

**5. Questions.** Besides the famous questions indicated in Section 1 we also ask:

**5.1.** Must $(E, p+q)$ be barrelled if $p$ and $q$ are non-comparable complete norms for a vector space $E$ (cf. 3.6)? **(P 1017)**

**5.2.** Must a countable codimensional subspace of a Banach space have a smaller complete norm (cf. 3.3 and [13])? **(P 1018)**

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