COLLOQUIUM MATHEMATICUM

LI DÉDIÉ À M. STANISŁAW HARTMAN 1987

CONJUGATION-INVARIANT MEANS

BY

V. LOSERT AND H. RINDLER (WIEN)

Let G be a locally compact group with left Haar measure dx and unit element e. For $x \in G$, the corresponding inner automorphism (*conjugation*) induces a mapping τ'_x on $L^\infty(G)$ by $\tau'_x f(y) = f(xyx^{-1})$. The adjoint map τ_x on $L^1(G)$ is given by $\tau_x u(y) = u(x^{-1}yx)\Delta(x)$ (where Δ denotes the Haar modulus of G). A non-negative linear functional M on $L^\infty(G)$ satisfying $M(1) = 1$ (where on the left-hand side 1 denotes the function with constant value 1) is called a *mean* (see [6]).

Definition. 1) A mean M on $L^\infty(G)$ is called *conjugation-invariant* (c.i.), if $M(\tau'_x f) = M(f)$ for all $x \in G$, $f \in L^\infty(G)$. (In [4] Effros uses the term "inner-invariant".)

2) A net (u_α) in $L^1(G)$ is called *asymptotically central* (a.c.), if

$$\lim_\alpha \frac{\|\tau_x u_\alpha - u_\alpha\|_1}{\|u_\alpha\|_1} = 0 \quad \text{for all } x \in G.$$

(We assume that $u_\alpha \neq 0$ and put $\|u\|_1 = \int_G |u(y)|\,dy$.)

Recall that the existence of non-trivial central elements in $L^1(G)$ is equivalent to the existence of a compact, conjugation-invariant neighbourhood of the identity in G ([10]). This produces simple examples of c.i. means. A.c. approximate units and a certain subclass of c.i. means were studied in [8]. We show that the existence of a c.i. mean is equivalent to the existence of an a.c. net (Proposition 1). If G is amenable, then there exists a (non-unique) c.i. mean (Proposition 2). If G is connected, then the converse holds, i.e. existence of a c.i. mean implies amenability (Theorem 1).

In the case of discrete groups, δ_e (Dirac measure at e) furnishes a c.i. mean. Further examples come from finite conjugacy classes. If G has Kazhdan's property T, then all c.i. means arise in this way (Theorem 2, see also [1]). Other conditions for uniqueness were discussed earlier in [4] in the context of the property Γ of the associated von Neumann algebra (see Proposition 3).

PROPOSITION 1. *The following assertions are equivalent:*

(i) *There exists a conjugation-invariant mean on* $L^\infty(G)$.

(ii) *There exists an asymptotically central net* (u_α) *in* $L^1(G)$.

(iii) *There exists a net* (v_α) *in* $L^1(G)$ *such that* $v_\alpha \geqslant 0$, $\|v_\alpha\|_1 = 1$ *and* $\lim_\alpha \|\tau_x v_\alpha - v_\alpha\|_1 = 0$ *for all* $x \in G$.

Proof. (ii) $\Rightarrow$ (iii): Put $v_\alpha(x) = \dfrac{|u_\alpha(x)|}{\|u_\alpha\|_1}$.

(iii) $\Rightarrow$ (i) $\Rightarrow$ (ii): The proof of this is similar to [8], Theorem 2. If $(v_\alpha) \subseteq L^1(G) \subseteq L^\infty(G)'$ is given as in (iii), then any w^*-cluster point M in $L^\infty(G)'$ is a c.i. mean. Conversely, given a c.i. mean M, it can be approximated in the w^*-sense by a net (u_α) in $L^1(G)$ with $u_\alpha \geqslant 0$, $\|u_\alpha\|_1 = 1$. It follows that $w^*\text{-}\lim(\tau_x u_\alpha - u_\alpha) = 0$ for all $x \in G$. The w^*-topology induces the weak topology on $L^1(G)$ and, since for convex sets the weak closure coincides with the norm closure, we can replace (u_α) by some convex combinations to get $\lim \|\tau_x u_\alpha - u_\alpha\|_1 = 0$.

Remark. In the discrete case, a similar result was shown in [4]. The conditions (ii) and (iii) can be generalized to $L^p(G)$ $(1 \leqslant p < \infty)$ (compare [6], p. 46). By some manipulations it is possible to achieve $\lim_\alpha \|\tau_x v_\alpha - v_\alpha\|_1 = 0$ uniformly in x on compact subsets of G.

PROPOSITION 2. *If G is amenable, then there exists a conjugation-invariant mean. This mean is not unique unless $G = \{e\}$.*

Proof. Any mean on $L^\infty(G)$ that is invariant under left and right translations is clearly c.i. Such means exist if G is amenable by [6], p. 29. On the other hand, it was shown in [8] Theorem 3 that if G is amenable, there exists a c.i. mean on $L^\infty(G)$ which coincides with δ_e for bounded continuous functions.

Remark. Regarding uniqueness, the situation is slightly different from that in the case of translation-invariant means. If G is amenable as a discrete group, then by results of Granirer and Rudin the translation-invariant mean is not unique ([6], p. 91, [12]). But e.g. in the case of $G = SO(n)$ $(n \geqslant 5)$ (or more generally when G has a dense subgroup, satisfying Kazhdan's property T), the left invariant mean is unique ([9]).

THEOREM 1. *Let G be a connected locally compact group. Then there exists a conjugation-invariant mean on $L^\infty(G)$ iff G is amenable.*

Remark. This result has been announced in [7].

Proof. One direction follows from Proposition 2. Now assume that there exists a c.i. mean M on $L^\infty(G)$ and that G is not amenable. If H is a closed normal subgroup of G, then $L^\infty(G/H)$ is embedded into $L^\infty(G)$ and M induces a c.i. mean on $L^\infty(G/H)$. By Yamabe's theorem [11], Theorem 4.6, there exists a closed normal subgroup K of G such that $G_1 = G/K$ is a Lie group. Let R be the radical of G_1 (i.e. the maximal solvable normal subgroup of G). By [6], p. 53, G_1/R is a non-compact semi-simple Lie group, it is connected and has trivial center (by the maximality of R). Hence it is sufficient to consider the case where G is a connected semi-simple Lie group with trivial center. We will show that if G is not compact, then property (iii) of Proposition 1 cannot hold.

Let $J_1, \ldots, J_r$ be a maximal system of pairwise non-conjugate Cartan subgroups of G. These are abelian, since G has trivial center [13], I. 1.4.1.5, p. 111. Since G is unimodular, each of the coset spaces G/J_i $(1 \leqslant i \leqslant r)$ carries a measure $d\dot{z}$ that is invariant under $L_x g(\dot{z}) = g\big((x^{-1}z)^{\cdot}\big)$ $(x \in G)$. Here we write $\dot{z} = zJ_i$. By [13], II. 8.1.2, p. 66, we have for $f \in L^1(G)$

(1) $$\int_G f(z)\,dz = \sum_{i=1}^{r} \int_{J_i} w_i(y) \int_{G/J_i} f(zyz^{-1})\,d\dot{z}\,dy.$$

(Since J_i is abelian, zyz^{-1} depends only on the left coset $\dot{z} = zJ_i$ of z; $w_i \geqslant 0$ signifies some weight function.) If property (iii) of Proposition 1 holds, then the following is true:

(2) Given $\varepsilon > 0$ and a finite subset F of G, there exists $u \in L^1(G)$ with $u \geqslant 0$, $\|u\|_1 = 1$ such that $\sum_{x \in F} \|\tau_x u - u\|_1 < \varepsilon \|u\|_1$.

From (1), (2) we get (recall that G is unimodular):

(3) $$\sum_{i=1}^{r} \sum_{x \in F} \int_{J_i} w_i(y) \int_{G/J_i} |u(x^{-1}zyz^{-1}x) - u(zyz^{-1})|\,d\dot{z}\,dy < \varepsilon \sum_{i=1}^{r} \int_{J_i} w_i(y) \int_{G/J_i} u(zyz^{-1})\,d\dot{z}\,dy.$$

Hence for some $i \in \{1, \ldots, r\}$ and some $y \in J_i$, we have:

(4) $$\sum_{x \in F} \int_{G/J_i} |u(x^{-1}zyz^{-1}x) - u(zyz^{-1})|\,d\dot{z} < \varepsilon \int_{G/J_i} u(zyz^{-1})\,d\dot{z} < \infty.$$

Put $g(\dot{z}) = u(zyz^{-1})$. Then $g \in L^1(G/J_i)$ and (4) implies

(5) $$\|L_x g - g\|_1 < \varepsilon \|g\|_1 \quad \text{for all } x \in F.$$

(Where $\|\ \|_1$ refers to the measure $d\dot{z}$ on G/J_1.)

Since the pairs (ε, F) form a directed set and there are only finitely many values of i, it is easy to see that the index i in (5) can be chosen independently of $\varepsilon > 0$ and the finite subset F of G. By [5], p. 28, this implies that G/J_i is amenable. But since J_i is abelian (hence amenable), it would follow that G is amenable ([5], p. 16). This is a contradiction if G is not compact.

Recall that a group G satisfies *Kazhdan's property T* if the trivial representation is isolated in the unitary dual $G^{\wedge}$ of G. G is said to be an *ICC-group*, if all non-trivial conjugacy classes are infinite.

THEOREM 2. *If G is a discrete group satisfying Kazhdan's property T, then any conjugation-invariant mean on $L^\infty(G)$ belongs to the w^*-closure of the center of $L^1(G)$. In particular, if in addition G is an ICC-group, then δ_e is the unique conjugation-invariant mean.*

Proof. Let M be a c.i. mean. As described in the proof of Proposition 1, we get a net $(u_\alpha) \subseteq L^1(G) \subseteq L^\infty(G)'$ such that $M = w^*\text{-}\lim u_\alpha$, $u_\alpha \geqslant 0$, $\|u_\alpha\|_1 = 1$ and $\lim \|\tau_x u_\alpha - u_\alpha\|_1 = 0$ for all $x \in G$. Put $v_\alpha = u_\alpha^{1/2}$; then we have $\lim \|\tau_x^{(2)} v_\alpha - v_\alpha\|_2 = 0$ for all $x \in G$, where $\tau_x^{(2)} v(y) = v(x^{-1} yx)$ is the unitary representation on $L^2(G)$ induced by the inner automorphisms (compare [6], p. 46). Write $v_\alpha = v'_\alpha + v''_\alpha$, where v'_α belongs to the subspace M of $L^2(G)$ where $\tau^{(2)}$ acts trivially and $v''_\alpha \in M^\perp$. Then $\lim \|\tau_x^{(2)} v''_\alpha - v''_\alpha\|_2 = 0$. If $c = \limsup \|v''_\alpha\|_2 > 0$, then for some subset of (v''_α) we get $(\tau_x^{(2)} v''_\alpha, v''_\alpha) \to c^2$ for all $x \in G$ (where (,) denotes the inner product on $L^2(G)$). It would follow that $\tau^{(2)}|M^\perp$ contains the trivial representation weakly ([3], 3.4.10, p. 68), hence by property T, $\tau^{(2)}|M^\perp$ would contain the trivial representation strongly, contrary to the definition of M. Thus $c = 0$, i.e. $\lim \|v_\alpha - v'_\alpha\|_2 = 0$. Put $u'_\alpha = (v'_\alpha)^2$; then u'_α belongs to the center of $L^1(G)$ and $\lim \|u_\alpha - u'_\alpha\|_1 = 0$ ([6], p. 47). Consequently, $M = w^*\text{-}\lim u'_\alpha$.

EXAMPLES. $SL(n, Z)$ has property T for $n \geqslant 3$ ([9], p. 234). The center Z consists of the scalar matrices. If n is odd, Z is trivial, if n is even, it has order 2. No other finite conjugacy classes do exist. Hence, if n is odd, δ_e is the unique c.i. mean. If n is even, the same holds for $PSL(n, Z) = SL(n, Z)/Z$.

Remarks. Discrete groups which have a c.i. mean different from δ_e were called *inner amenable* in [4]. A related result was shown in [1], Theorem 11.

PROPOSITION 3. *Let G be a discrete group which is the free product of groups H_1 and H_2, where H_1 has at least two and H_2 at least three elements. Then δ_e is the unique conjugation-invariant mean.*

Proof. This is essentially contained in [4]. We use the idea of von Neumann to construct "paradoxical" decompositions. Take $a \in H_1 \setminus \{e\}$, $b, c \in H_2 \setminus \{e\}$ with $b \neq c$. Let D be the set of elements of G whose representation as a reduced word starts with an element of H_1. Then $G = D \cup aDa^{-1} \cup \{e\}$ and D, bDb^{-1}, cDc^{-1} are disjoint. As shown in [4] this implies that any c.i. mean is supported by $\{e\}$.

Remark. For a free group with at least two generators this was established by a different method in [2]. If $H_1 = H_2 = Z_2$, the free product is solvable, hence by Proposition 2 the c.i. mean is not unique.

EXAMPLE. *$PSL(2, Z) = SL(2, Z)/Z$ is the free product of Z_2 and Z_3.*

REFERENCES

[1] C. A. Akemann and M. E. Walter, *Unbounded negative definite functions*, Canadian Journal of Mathematics 33 (1981), p. 862–871.

[2] M. Bożejko, *Some aspects of harmonic analysis on free groups*, Colloquium Mathematicum 41 (1979), p. 265–271.

[3] J. Dixmier, *Les C*-algèbres et leurs représentations*, Gauthier-Villars, Paris 1964.

[4] E. G. Effros, *Property Γ and inner amenability*, Proceedings of the American Mathematical Society 47 (1975), p. 483–486.

[5] P. Eymard, *Moyennes Invariantes et Représentations Unitaires*, Lecture Notes in Mathematics 300, Springer, Berlin–Heidelberg–New York 1972.

[6] F. P. Greenleaf, *Invariant means on topological groups*, van Nostrand, Mathematical Studies, New York 1969.

[7] M. Grosser, V. Losert and H. Rindler, *"Double multipliers" und asymptotisch invariante approximierende Einheiten*, Österreichische Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse. Anzeiger, 1980, p. 7–11.

[8] V. Losert and H. Rindler, *Asymptotically central functions and invariant extensions of Dirac measure*, p. 368–378 in: *Proceedings of the Conference on Probability Measures on Groups, Oberwolfach 1983*, Lecture Notes in Mathematics 1064, Springer, Berlin–Heidelberg–New York.

[9] G. A. Margulis, *Some Remarks on Invariant Means*, Monatshefte für Mathematik 90 (1980), p. 233–235.

[10] R. D. Mosak, *Central functions in group algebras*, Proceedings of the American Mathematical Society 29 (1971), p. 613–616.

[11] D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, New York 1955.

[12] W. Rudin, *Invariant means on L^∞*, Studia Mathematica 44 (1972), p. 219–227.

[13] G. Warner, *Harmonic Analysis on Semi-Simple Lie Groups I, II*, Springer, Berlin–Heidelberg–New York 1972.

INSTITUT FÜR MATHEMATIK
UNIVERSITÄT WIEN
WIEN, AUSTRIA

Reçu par la Rédaction le 20. 01. 1984