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ON THE EXISTENCE OF LINEAR CONNECTION VALUED CONCOMITANTS OF GEOMETRIC OBJECTS

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Let $P \to M$ be a principal H-bundle over a paracompact differentiable manifold M of dimension n and let ϱ be a left action of the Lie group H on R^N. A geometric object τ of type ϱ on P is an arbitrary smooth equivariant mapping τ: $P \to R^N$. Thus $\tau(pa) = \varrho(a^{-1})\tau(p)$ or, simply, $\tau(pa) = a^{-1}\tau(p)$ for each $a \in H$, $p \in P$. A geometric object τ will be called *0-deformable* if H acts transitively on $\operatorname{Spa}\tau = \tau(P)$. This property is equivalent to the existence of a subbundle Q of P on which τ is constant. If $\tau|_a = \mathring{\tau}$ and $G_{\mathring{\tau}}$ denotes the isotropy group of $\mathring{\tau}$, then Q is a principal $G_{\mathring{\tau}}$-bundle over M.

A geometric object λ on P will be called a *concomitant* of τ if there exists an H-equivariant map f: $\operatorname{Spa}\tau \to \operatorname{Spa}\lambda$ such that $\lambda = f \circ \tau$. Note that we do not require f to be smooth although τ and λ are smooth.

LEMMA. *Given τ and λ such that τ is 0-deformable, λ is a concomitant of τ if and only if there exists a subbundle Q of P on which both τ and λ are constant and $G_\tau \subset G_\lambda$.*

Proof. The necessity is obvious. For the sufficiency, assume $\tau(p) = \tau(p')$. There are q, $q' \in Q$ and a, $a' \in H$ such that $p = qa$ and $p' = q'a'$. It follows that $a'a^{-1} \in G_{\tau(q)}$ and, therefore, $a'a^{-1} \in G_{\lambda(q)}$. Hence $a'a^{-1}_{\lambda(q)} = \lambda(q) = \lambda(q')$, which implies $\lambda(p) = \lambda(p')$. Thus the equality $f(\tau) = f(\tau(p)) := \lambda(p)$ well defines a map f: $\operatorname{Spa}\tau \to \operatorname{Spa}\lambda$, which is obviously equivariant.

Existence of a symmetric linear τ-connection. Let τ be a linear geometric object on the frame bundle $L^1 M$ of type ϱ. Then ϱ is a linear representation of L_n^1 on R^N. Denote by ϱ' the induced representation of the Lie algebra $\mathrm{gl}(n)$ on R^N. The first holonomic prolongation $j^1\tau = (\tau, \partial\tau)$ is a linear geometric object on the 2-frame bundle $L^2 M$. If

$$(\bar{\tau}, \partial\bar{\tau}) = (a, \xi)\cdot(\tau, \partial\tau) \quad \text{and} \quad (a, \xi) = (a_j^i, \xi_{kj}^i) \in L_n^2,$$

then

$$\bar{\tau} = \varrho(a)\tau, \quad \partial_{\bar{k}}\bar{\tau} = a_{\bar{k}}^{k}\varrho(a)[\varrho'(a^{-1}\xi_k)\tau + \partial_k\tau], \tag{1}$$

where ξ_k is the matrix ξ_{kj}^i, where k is fixed, and $\partial_k = \partial/\partial x^k$.

Given a linear connection object $\Gamma = (\Gamma^i_{jk})$ (from now on called shortly a *connection*), the convariant derivative of τ with respect to Γ is

$$\nabla_k \tau = \partial_k \tau + \varrho'(\Gamma_k)\tau. \tag{2}$$

A linear connection object Γ is said to be a *τ-connection* if $\nabla_k \tau = 0$.

It is well known that there exists a τ-connection if and only if τ is 0-deformable (cf. [2]). We know also that there exists a torsionless connection in a G-structure P if and only if the first structure tensor of P vanishes [2]. We shall give the condition of the existence of a symmetric τ-connection directly in terms of τ and its first derivatives.

Definition. A 0-deformable geometric object τ on $L^1 M$ will be called 1-*flat* if for each $x \in M$ there exists a 2-frame $p \in L^2 M$ at which $\partial_k \tau = 0$ $(k = 1, \ldots, n)$. If so, fixing $\mathring{\tau} \in \operatorname{Spa}\tau$ and changing suitably the first order part of p, we can get a 2-frame p at which $\tau = \mathring{\tau}$ and $\partial_k \tau = 0$. Therefore, τ is 1-flat if and only if $j^1 \tau$ is 0-deformable and $(\mathring{\tau}, 0) \in \operatorname{Spa} j^1 \tau$.

THEOREM 1. *A symmetric linear τ-connection exists if and only if τ is 1-flat.*

Proof. Γ is a geometric object on $L^2 M$ subject to the action of elements $(a, \xi) \in L^2_n$ by

$$\bar{\Gamma} = \operatorname{ad}_{a^{-1}} \Gamma + a^{-1} \xi. \tag{3}$$

Γ being symmetric, in the normal coordinate system we have $\Gamma = 0$. If $\nabla_k \tau = 0$, putting $\Gamma = 0$ into (2) we obtain $\partial_k \tau = 0$.

Conversely, let τ be 1-flat. There exists a subbundle Q of $L^2 M$ on which $j^1 \tau$ takes a constant value $(\mathring{\tau}, 0)$. Let $\{U_\alpha\}$ be an open covering of M such that for each α there exists a section $\sigma_\alpha \colon U_\alpha \to Q$. On U_α we define a connection $\overset{\alpha}{\Gamma}$ putting $\overset{\alpha}{\Gamma} = 0$ on σ_α. Then $\overset{\alpha}{\Gamma}$ is a symmetric τ-connection on U_α. Using the standard procedure of overlapping $\{\overset{\alpha}{\Gamma}\}$ by a smooth partition of unity inscribed in $\{U_\alpha\}$, we get a symmetric τ-connection on M.

Remark. There may not exist a Riemannian τ-connection even if τ is integrable (see Example 6.9 on p. 164 of [4], concerning a complex manifold which does not admit any Kaehler metric).

Existence of connection valued concomitants. The above problem for particular tensorial structures has been widely considerd in the literature. Recently, the authors of [1] gave a negative answer in the case of almost complex, almost product, and almost tangent structures as well as for the structure $f^3 + f = 0$, f being a (1, 1) tensor field. We try to solve it generally.

Recall that if g is a linear Lie algebra of $(n \times n)$-matrices, then the first prolongation of g is defined to be

$$g^{(1)} = g \otimes R^{n*} \cap R^n \otimes S^2(R^{n*})$$

(see [5]).

THEOREM 2. *Let τ be a linear geometric object on $L^1 M$. If there exists a connection valued concomitant $\Gamma(j^1\tau)$, then $g^{(1)}=0$, where g is the Lie algebra of the isotropy group of an arbitrarily chosen element of* $\operatorname{Spa}\tau$.

Proof. Let Γ be a concomitant of $j^1\tau$. Without loss of generality we may assume that Γ is symmetric. By (3), in any frame $p\in L^2 M$ in which $\Gamma=0$ the isotropy group of Γ is $(L_n^1, 0)$. Obviously, the isotropy group G of $j^1\tau$ in p is contained in $(L_n^1, 0)$. Take $\xi\in g^{(1)}$, where g is calculated for τ at the first order part of p. Substituting $(e, \xi)\in L_n^2$ into (1) (e is the unit matrix), we see that (e, ξ) is in G. This implies $\xi=0$. Thus $g^{(1)}=0$. Since in other frames of the same fibre of $L^2 M$ the isotropy groups of $j^1\tau$ are conjugated with G, we obtain $g_\tau^{(1)}=0$ for any value of τ.

THEOREM 3. *Let τ be a 0-deformable linear geometric object on $L^1 M$ such that $g^{(1)}=0$ and assume that one of the following cases holds:*

(i) *τ is 1-flat,*

(ii) *the homogeneous space L_n^1/G, G being isotropy group of τ, is reductive.*

Then there exists a connection valued concomitant of $j^1\tau$.

Proof. (i) By Theorem 1 there exists a symmetric τ-connection. It satisfies the equations

$$\partial_k\tau+\varrho'(\Gamma_k)\tau=0 \quad (k=1,\ldots,n). \tag{4}$$

If Γ' is another symmetric τ-connection, then $\Gamma'-\Gamma\in g^{(1)}=0$. Thus equations (4) determine the symmetric τ-connection uniquely. Consequently, if $j^1\tau(p)=j^1\tau(p')$, then $\Gamma(p)=\Gamma(p')$. Therefore, there exists an L_n^2-equivariant mapping $f\colon \operatorname{Spa} j^1\tau\to\operatorname{Spa}\Gamma$ such that $\Gamma=f\circ j^1\tau$.

(ii) Let Q be a subbundle of L_n^1 on which τ takes a constant value $\mathring{\tau}$. Let G and g be calculated for $\mathring{\tau}$. By assumption, $\mathrm{gl}(n)$ admits a subspace m such that $gl(n)=g\oplus m$ and $\mathrm{ad}_G m=m$. Thus

$$\mathrm{gl}(n)\otimes R^{n*}=g\otimes R^{n*}\oplus m\otimes R^{n*}=A\oplus B.$$

Let ad_a stand for the induced adjoint representation of L_n^1 on $\mathrm{gl}(n)\otimes R^{n*}$. In a frame $p=qa$, $q\in Q$, $a\in L_n^1$, we have

$$\mathrm{gl}(n)\otimes R^n=\mathrm{ad}_{a^{-1}}A\oplus\mathrm{ad}_{a^{-1}}B=A(p)\oplus B(p)$$

independently of the choice of q. Note that $A(p)$ coincides with $g_{\tau(p)}\otimes R^{n*}$.

Consider any τ-connection Γ. Equations (4) determine Γ modulo an element of $A(p_1)$, where p_1 is the $L^1 M$-projection of the corresponding 2-frame. Thus the $B(p_1)$-component $\tilde{\Gamma}$ of Γ is determined uniquely by (4) and $\tilde{\Gamma}$ satisfies (4). If $\bar{p}=p(a,\xi)$, then $\Gamma(\bar{p})=\mathrm{ad}_{a^{-1}}\Gamma(p)+a^{-1}\xi$. As $a^{-1}\xi$ is a symmetric element of $\mathrm{gl}(n)\otimes R^{n*}$ and $g_{(p_1)}^{(1)}=0$, $a^{-1}\xi$ must be in $B(p_1)$. Therefore,

$$\tilde{\Gamma}(\bar{p})=\mathrm{ad}_{a^{-1}}\tilde{\Gamma}(p)+a^{-1}\xi,$$

which means that $\tilde{\Gamma}$ has a connection-like transformation rule.

Finally, on any section $\sigma_1\colon U \to Q$ the subspaces A and B are constant. Thus, if $\sigma\colon U \to L^2 M$ covers σ_1, then $\tilde{\Gamma}(\sigma(x))$ is a smooth projection of $\Gamma(\sigma(x))$. It follows that $\tilde{\Gamma}$ is a smooth connection object on $L^2 M$. $\tilde{\Gamma}$ is also a τ-connection. Since equations (4) determine $\tilde{\Gamma}$ uniquely in any 2-frame, the same argument as in the proof of (i) leads to the conclusion that $\tilde{\Gamma}$ is a concomitant of $j^1 \tau$. This completes the proof.

Application. A linear group G is said to be *involutive* if the Spencer cohomology groups $H^{k,l}(g)$ vanish for all l and $k > 0$ (cf. [4] and [5]). In particular, if

$$H^{1,2}(g) := \operatorname{Ker}(g \otimes \Lambda^2 R^{n*} \xrightarrow{\delta} R^n \otimes \Lambda^3 R^{n*})/\delta(g^{(1)} \otimes R^{n*}) = 0,$$

then $g^{(1)} \neq 0$ whenever g is non-zero. Hence

Corollary. *If the isotropy groups of a linear geometric object τ on $L^1 M$ are involutive and have a non-zero Lie algebra, then there is no connection valued concomitant of τ and its first derivatives.*

Since arbitrary tensorial structure of type (1, 1) as well as the almost symplectic structure and the almost multifoliate structure involve involutive isotropy groups with non-trivial Lie algebra, none of them gives rise to a connection valued concomitant of first order.

The known positive cases are the Riemannian structure, the trivial structure (with a global field of 1-frames), and the almost quaternionic structure (cf. [6]). The last one is based on three (1, 1) tensor fields.

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