COLLOQUIUM MATHEMATICUM

VOL. XXXVII 1977 FASC. 1

A CHARACTERIZATION OF LIGHT OPEN MAPPINGS AND THE EXISTENCE OF GROUP ACTIONS

BY

LOUIS F. McAULEY (BINGHAMTON, NEW YORK)

1. Introduction. The rather remarkable work of Wilson [13] and [14], and Walsh [8]-[11] provide new insights into the behavior of open mappings on manifolds and, in particular, that of light open mappings. Wilson has answered affirmatively the questions raised by Steenrod (see [2], [4], Problems 41 and 42). These results show the existence of the various monotone and light open mappings some of which raise dimension and some of which have Cantor sets as point inverses.

Here we give a characterization of light open mappings in terms of a sequence of special coverings. The question of what open mappings are topologically equivalent to orbit mappings of actions by (topological) groups has been of interest for some time.

Fintushel has shown in his thesis [3] that if $f\colon M^n \Rightarrow N^{n-1}$ ($n = 3$ or 4) is a PL open mapping from the closed and connected PL manifold M^n onto another N^{n-1} such that $f^{-1}f(x)$ is either a point or homeomorphic to S^1 (a circle), then f is equivalent to the orbit mapping of a local action S^1 on M^n. Little progress has been made on this problem when f is a *light open mapping*, i.e., $f^{-1}f(x)$ is totally disconnected (and f is open). We give a result for certain finite-to-one open mappings on S^2 (a 2-sphere). The techniques are useful in obtaining other results. We obtained this result prior to the announcement of the work of Edmonds [1]. His work is related to ours but does not include it.

2. Characterization of light open mappings. We prove a theorem which characterizes light open mappings defined on Peano continua in terms of a special sequence of coverings. A proof of the sufficiency appears in [6]. The proof of the necessity is in part due to Eric Robinson.

Definition (Property M). Suppose that $\{C_j = C_1^j, C_2^j, \ldots, C_{n_j}^j\}$ is a finite closed covering of a space X. We say that C_j has *Property* M if and only if, for each $i = 1, 2, \ldots, n_j$,

(1) C_i^j is a finite collection $\{C_{i,1}^j, C_{i,2}^j, \ldots, C_{i,k_i}^j\}$ of pairwise disjoint closed subsets of X,

(2) $\operatorname{diam} C_{i,s}^j < 1/j$ for $1 \leqslant s \leqslant k_i$,

(3) if $(C_s^j)^* \cap (C_t^j)^* \neq \emptyset$ (where G^* denotes the union of the elements of G), then, given u with $1 \leqslant u \leqslant k_s$, there is v, $1 \leqslant v \leqslant k_t$, such that $C_{s,u}^j \cap C_{t,v}^j \neq \emptyset$, and

(4) each $x \in X$ is contained in the interior of some element of C_k^i for each i.

Definition (strong refinement). Suppose that

$$C_i = \{C_1^i, C_2^i, \ldots, C_{n_i}^i\} \quad \text{and} \quad C_j = \{C_1^j, C_2^j, \ldots, C_{n_j}^j\}$$

are two finite closed coverings of X, where $C_s^t = \{C_{s,1}^t, C_{s,2}^t, \ldots, C_{s,k_s}^t\}$, $1 \leqslant s \leqslant n_t$, and $t = i$ or j. We say that C_j *strongly refines* C_i if and only if

(1) each element of C_s^j lies in some element of C_t^i for some t, and

(2) if some element of C_s^j lies in an element of C_t^i, then each element of C_s^j lies in some element of C_t^i.

THEOREM 1 (McAuley and Robinson). *Suppose that each of X and Y is a compact, connected, and locally connected metric space and that f is a (continuous) mapping of X onto Y ($f\colon X \Rightarrow Y$). Then f is light and open if and only if there is a sequence $\{C_i\}$ of closed covers of X such that*

(a) *for each i, $C_i = \{C_1^i, C_2^i, \ldots, C_{n_i}^i\}$ has Property* M,

(b) *C_{i+1} strongly refines C_i,*

(c) *$f(C_{n,k}^i) = f(C_{n,q}^i)$, where $C_s^i = \{C_{s,1}^i, C_{s,2}^i, \ldots, C_{s,n_s}^i\}$,*

(d) *each $C_{s,t}^i$ can be taken to be connected (or a Peano continuum with non-empty interior), and*

(e) *for each $y \in Y$,*

$$f^{-1}(y) = \bigcap_{i=1}^{\infty} (C_{s_i}^i)^*$$

for some nested sequence $\{(C_{s_i}^i)^*\}$, where $C_{s_i}^i \in C_i$, and, conversely, each such nested sequence $\{(C_{s_i}^i)^*\}$ has the property that, for some y,

$$f^{-1}(y) = \bigcap_{i=1}^{\infty} (C_{s_i}^i)^* \quad \textit{and} \quad \operatorname{Int}(C_{s_i}^i)^* \supset f^{-1}(y).$$

We shall need the following theorems in our proof of the necessity of the special sequence of coverings.

THEOREM A ([12], p. 148). *If $f\colon X \Rightarrow Y$ is an open mapping, where each of X and Y is a Peano space (compact locally connected metric space), and R is a connected open subset of Y, then $f^{-1}(R)$ has at most a finite number of components each of which maps onto R under f.*

THEOREM B ([12], p. 131). *If $f: X \Rightarrow Y$ is a light mapping, where each of X and Y is a compact metric space, and $\varepsilon > 0$, then there is a $\delta > 0$ such that, for each closed and connected subset C of Y with diameter less than δ, each component of $f^{-1}(C)$ has diameter less than ε.*

THEOREM C ([12], p. 189). *If $f: X \Rightarrow Y$ is a light open mapping, where each of X and Y is a Peano continuum, and K is a Peano continuum in Y whose interior is dense in K, then $f^{-1}(K)$ is locally connected (and each component is a Peano continuum).*

Proof of Theorem 1. By Theorem B, there is a $\delta_1 > 0$ such that if C is a closed connected subset of Y with diameter less than δ_1, then each component of $f^{-1}(C)$ has diameter less than 1. Let $\varepsilon_1 = \min[\delta_1, 1]$, and let $O_1 = \{O_1^1, O_2^1, \ldots, O_{n_1}^1\}$ be a closed covering of Y such that

(a) for each i, O_i^1 is the closure of a non-empty uniformly locally connected and connected subset of Y (and is, therefore, a Peano continuum),

(b) each $y \in Y$ lies in the interior of a member of O_1,

(c) $\operatorname{diam} O_i^1 < \varepsilon_1$, and

(d) $O_i^1 \not\supset O_j^1$ for $i \neq j$.

Now, let $C_i^1 = \{C_{i,1}^1, C_{i,2}^1, \ldots, C_{i,m_{1i}}^1\}$ be the finite set (using Theorem A) of components of $f^{-1}(O_i^1)$. Clearly, $f(C_{i,j}^1) = O_i^1$. Each $C_{i,j}^1$ is a Peano continuum (by Theorem C) with non-empty interior and has diameter less than 1. The collection C_1 of all C_i^1 $(1 \leqslant i \leqslant n_1)$ is a closed covering of X and each $x \in X$ is in the interior of an element of C_i^1 for some i.

Next, we show that C_1 has Property M. Suppose that $(C_s^1)^* \cap (C_t^1)^* \neq \emptyset$. Thus, there exist p, q with $1 \leqslant p \leqslant m_{1s}$ and $1 \leqslant q \leqslant m_{1t}$ such that $x \in C_{s,p}^1 \cap C_{t,q}^1$. Let u be given so that $1 \leqslant u \leqslant m_{1s}$. Since $f(C_{s,u}^1) = O_s^1$, there is a $y \in f^{-1}f(x) \cap C_{s,u}^1$. Since

$$(C_t^1)^* = f^{-1}(O_t^1) \supset f^{-1}f(x),$$

there is v, $1 \leqslant v \leqslant m_{1t}$, such that $y \in C_{t,v}^1$. Hence, $C_{s,u}^1 \cap C_{t,v}^1 \neq \emptyset$ and C_1 has Property M.

Next, by Theorem B, there is a $\delta_2 > 0$ such that, for each closed connected subset C of Y with diameter less than δ_2, each component of $f^{-1}(C)$ has diameter less than 1/2. Let $\varepsilon_2 = \min[\delta_2, 1/2]$, and let $O_2 = \{O_1^2, O_2^2, \ldots, O_{n_2}^2\}$ be a closed covering of X such that

(a) O_i^2 is the closure of a connected and uniformly locally connected open set (and hence a Peano continuum),

(b) each $x \in X$ is in the interior of some O_i^2,

(c) $\operatorname{diam} O_i^2 < \varepsilon_2$,

(d) $O_i^2 \not\supset O_j^2$ for $i \neq j$, and

(e) each $O_i^2 \subset O_k^1$ for some k (O_2 refines O_1).

Let $C_i^2 = \{C_{i,1}^2, C_{i,2}^2, \ldots, C_{i,m_{2i}}^2\}$ be the finite set of components of $f^{-1}(O_i^2)$. Clearly, $f(C_{i,j}^2) = O_i^2$, each $C_{i,j}^2$ is a Peano continuum with non-empty interior, and $\operatorname{diam} C^2_{i,j} < 1/2$. The collection C_2 of all $C^2_{i,j}$ $(1 \leqslant i \leqslant n_2)$ is a closed covering of X with Property M. It follows easily that C_2 strongly refines C_1.

In the manner described above, we define a sequence $\{C_k\}$ of closed coverings of X such that, for each k, C_{k+1} strongly refines C_k, and C_k may be partitioned into a finite number of collections C^k_i $(i = 1, 2, \ldots, n_k)$ such that

(a) $C^k_i = \{C^k_{i,1}, C^k_{i,2}, \ldots, C^k_{i,m_{ik}}\}$ is a finite collection of pairwise disjoint Peano continua each with non-empty interior,

(b) $C_k = \{C^k_1, C^k_2, \ldots, C^k_{n_k}\}$ has Property M,

(c) $f(C^k_{i,j}) = O^k_i$, where O^k_i is the closure of a uniformly locally connected and connected open set in Y,

(d) $f^{-1}(O^k_i) = (C^k_i)^*$,

(e) $\operatorname{diam} C^k_{i,j} < 1/k$, and

(f) $\operatorname{diam} f(C^k_{i,j}) < 1/k$.

Now, we show that for each $y \in Y$,

$$f^{-1}(y) = \bigcap_{i=1}^{\infty} (C^i_{s_i})^*$$

for some nested sequence $\{(C^i_{s_i})^*\}$, where $C^i_{s_i} \in C_i$ for each i. For each y and i, there is $C^i_{s_i} \in C_i$ such that $\operatorname{Int}(C^i_{s_i})^* \supset f^{-1}(y)$. Since C_{i+1} strongly refines C_i and each C_k has Property M, it follows that $(C^i_{s_i})^* \supset (C^{i+1}_{s_{i+1}})^*$. Thus,

$$\bigcap_{i=1}^{\infty} (C^i_{s_i})^* \supset f^{-1}(y).$$

Since

$$f(C^i_{s_i})^* = O^i_{p_i},$$

where

$$\operatorname{diam} O^i_{p_i} < \frac{1}{i}, \quad f^{-1}(O^i_{p_i}) = (C^i_{s_i})^* \quad \text{and} \quad y = \bigcap_{i=1}^{\infty} O^i_{p_i},$$

it follows that

$$f^{-1}(y) = \bigcap_{i=1}^{\infty} (C^i_{s_i})^*.$$

Finally, if $\{(C^i_{s_i})^*\}$ is a nested sequence with $C^i_{s_i} \in C_i$, then

$$\bigcap_{i=1}^{\infty} (C^i_{s_i})^* = f^{-1}(y) \quad \text{for some } y \in Y.$$

This is an easy consequence of the properties of C_i. Consequently, Theorem 1 is true.

3. Certain light-open mappings are equivalent to orbit mappings of actions by groups — an application of Theorem 1. First, we give a special case of a finite-to-one light open mapping on S^2 which is equivalent to the orbit mapping of an action Z_n on S^2.

THEOREM 2. *Suppose that* $f\colon S^2 \Rightarrow Y$ *is a light open mapping such that* $f^{-1}f(z)$ *consists of exactly* n *points* ($n > 1$) *except the two points* a *and* b, *where* $f^{-1}f(a) = a$ *and* $f^{-1}f(b) = b$. *Then there exists a periodic homeomorphism* h *of period* n *on* S^2 (*i.e.,* Z_n *acts on* S^2 *so that the orbit map* $\varphi\colon S^2 \Rightarrow S^2/Z_n \cong Y$ *is equivalent to* f).

Proof. From a theorem of [12], p. 197, it follows that Y is one of the following: a 2-sphere, a projective plane, or a 2-cell. Clearly, Y is not a 2-cell D, since f would not be a local homeomorphism at each point of $f^{-1}(\partial D)$ whereas the only points of S^2 at which f is not a local homeomorphism are the points a and b.

Suppose that Y is the projective plane P. We have $P = A \cup B$, where A is a 2-cell containing $f(a)$ and $f(b)$ in its boundary ∂A, and B is a Möbius band with boundary equal to ∂A such that the interior of A is disjoint from B. Consider a component C of $S^2 - f^{-1}(A)$ which maps onto $B - \partial B$. The simple closed curve ∂C is mapped homeomorphically by $f|\partial C$ onto $\partial B = \partial A$. Indeed, $f|(C \cup \partial C)$ is a light open mapping of $C \cup \partial C = \bar{C}$ (which is a 2-cell) onto B. This is impossible by [12], (i), p. 197. Thus, Y is a 2-sphere S^2.

Now, let α be any simple arc (homeomorph of $[0, 1]$) in Y with endpoints $f(a)$ and $f(b)$. From results of [12] it follows that $f^{-1}(\alpha)$ is a collection of n simple arcs $A_1, A_2, \ldots, A_n$ each with endpoints a and b such that $(A_i - \{a, b\}) \cap A_j = \emptyset$ for $i \neq j$. Furthermore, $f|A_i$ is a homeomorphism of A_i onto α. Each of the n components C_i ($i = 1, 2, \ldots, n$) of $S^2 - f^{-1}(\alpha)$ is an open 2-cell whose boundary is the union of a pair of the simple arcs $A_j \cup A_k$. Index the n components and simple arcs so that $\partial C_i = A_i \cup A_{i+1}$ except for C_n whose boundary is $A_n \cup A_1$. Again, from results of [12] it follows that $f|C_i$ is a homeomorphism of C_i onto $Y - \alpha$.

The indexing of the components C_i and simple arcs A_i as indicated above makes it possible to define a homeomorphism h of S^2 onto S^2 of period n as follows. If $x \in S^2 - \{a, b\}$, then either $x \in A_i$ or $x \in C_i$ for some i. Now, $f^{-1}f(x) = \{x_1, x_2, \ldots, x_n\}$, where either $x_i = f^{-1}f(x) \cap A_i$ or $x_i = f^{-1}f(x) \cap C_i$ depending on the cases $x \in A_i$ or $x \in C_i$. Clearly, $x = x_k$ for some $k = 1, 2, \ldots, n$. Let $h(x) = x_{k+1}$ except for $k = n$ in which case $h(x) = x_1$. Also, $h(a) = a$ and $h(b) = b$. It follows that h is a homeomorphism of S^2 onto S^2 which has the period n. Furthermore, Z_n acts on S^2 so that the orbit map $\varphi\colon S^2 \Rightarrow S^2/Z_n \cong Y$ is equivalent to f.

Alternate proof of Theorem 2. In the proof of Theorem 2 above, Theorem 1 is not applied. However, we shall outline a proof of this theorem with an application of Theorem 1 for the purpose of illustrating a technique which may be used to prove other theorems.

It is not difficult to obtain a sequence of closed coverings of S^2 which satisfy the hypothesis of Theorem 1. Consider certain triangulations T_i of $Y = S^2$ with the collection S_i of closed 2-simplexes in T_i such that

(1) if $\sigma \in S_i$, then $f^{-1}(\sigma) = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ is a collection of n 2-simplexes in S^2, where $f|\sigma_i$ is a homeomorphism and which are pairwise disjoint if and only if σ contains neither $f(a)$ nor $f(b)$,

(2) S_{i+1} is a subdivision of S_i, and

(3) $\max[\operatorname{diam}\sigma_k \mid f^{-1}(\sigma) = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ for all $\sigma \in S_i] < 1/i$.

For each i, let C_i denote the collection consisting of C_a^i which is the union of all $f^{-1}(\sigma)$ for $\sigma \in S_i$ with $f(a) \in \sigma$ and, similarly, of C_b^i plus all collections $C_k^i = \{\sigma_{k1}^i, \sigma_{k2}^i, \ldots, \sigma_{kn}^i\}$ for each $\sigma_k^i \in S_i$ with $a, b \notin \sigma_k^i$ and $f^{-1}(\sigma_k^i) = \{\sigma_{k1}^i, \sigma_{k2}^i, \ldots, \sigma_{kn}^i\}$. It follows that $\{C_i\}$ satisfies the assumptions of Theorem 1.

Consider a simple polygonal arc α from $f(a)$ to $f(b)$ consisting of 1-simplexes in T_i. Now, $f^{-1}(\alpha)$ is a collection of polygonal arcs $A_1, A_2, \ldots, A_n$ which are pairwise disjoint except for their common endpoints a and b. These can be indexed along with the components C_j of $S^2 - \bigcup_{k=1}^{n} A_k$, as in the first proof of Theorem 2, so that $\partial C_i = A_i \cup A_{i+1}$ with $\partial C_n = A_n \cup A_1$. Now, we can define a simplicial homeomorphism of period n on the nerve $N(C_i)$ in a manner similar to the way we defined h on S^2 in that proof. Thus, it is not difficult to obtain an inverse system of nerves of coverings of S^2, namely,

$$N(C_1) \xleftarrow{s_1} N(C_2) \xleftarrow{s_2} N(C_3) \xleftarrow{s_3} \cdots$$

(with s_i the obvious simplicial mappings, where $s_i h_{i+1}(x) = h_i s_i(x)$), and an inverse system of cyclic groups $X_n = G_i$ for each i,

$$G_1 \xleftarrow{\theta_1} G_2 \xleftarrow{\theta_2} G_3 \xleftarrow{\theta_3} \cdots,$$

where the homomorphism $\theta_i \colon G_{i+1} \Rightarrow G_i$ is induced by the simplicial mappings s_i. The inverse limit Z_n of the groups G_i provides an action on S^2, the inverse limit of the nerves $N(C_i)$. This motivates the following theorem which is almost obvious.

THEOREM 3. *Suppose that $f \colon X \Rightarrow Y$ is a light open mapping, where each of X and Y is a Peano continuum. Suppose also that $\{C_i\}$ is a sequence of closed coverings satisfying the assumptions of Theorem* 1. *Furthermore, for each i, there is a finite group G_i which acts simplicially on the nerve $N(C_i)$ of C_i such that*

(1) *the orbit of each vertex of $N(C_i)$ is exactly the set of vertices corresponding to a collection C_k^i of C_i for some k (described in Theorem* 1), *and*

(2) *each collection* C_j^i *of* C_i *corresponds to a set of vertices which is the orbit of a vertex under the action by* G_i.

There are simplicial mappings $s_i\colon N(C_{i+1}) \Rightarrow N(C_i)$ *and homomorphisms* $\theta_i\colon G_{i+1} \Rightarrow G_i$ *such that, for each* $x \in N(C_{i+1})$ *and* $g \in G_{i+1}$,

$$s_i g(x) = \theta_i(g)\big(s_i(x)\big).$$

Then the mapping f *is equivalent to the orbit mapping* φ *of a (topological) group* G, *the inverse limit* G_i *which acts on* X. *That is, there is* $\varphi\colon X \Rightarrow X/G \cong Y$ *and each orbit of a point* x *under* G *is* $f^{-1}f(x)$.

Indication of proof. Clearly, $X = \varprojlim N(C_i)$ and $\varprojlim G_i = G$ is a topological group.

The characterization of f by $\{C_i\}$ yields that, for each $y \in Y$,

$$f^{-1}(y) = \bigcap_{i=1}^{\infty} (C_{s_i}^i)^*$$

for some nested sequence $\{(C_{s_i}^i)^*\}$, where $C_{s_i}^i \in C_i$. Each $C_{s_i}^i$ is a finite collection of pairwise disjoint closed sets with non-empty interiors and corresponds to an orbit of a vertex of $N(C_i)$ under G_i. Thus, $G = \varprojlim G_i$ acts on $X \cong \varprojlim N(C_i)$, and $f^{-1}(y)$ is an orbit of a point in X. The various properties of $\{C_i\}$ along with the simplicial mappings s_i and homomorphisms θ_i insure that each $g \in G$ is indeed a homeomorphism of X onto X.

The proof follows easily.

QUESTIONS. Is it possible to obtain a proof of Theorem 2 by using directly the characterization of f as given by Theorem 1? (**P 998**)

Does the characterization of f as in Theorem 1 provide covering sequences $\{C_i\}$ with either "nice" nerves $N(C_i)$ or nerves containing subpolyhedra homeomorphic to X which approximate $N(C_i)$ nicely? (**P 999**)

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STATE UNIVERSITY OF NEW YORK AT BINGHAMTON
BINGHAMTON, NEW YORK

Reçu par la Rédaction le 29. 4. 1975;
en version modifiée le 20. 1. 1976