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ON FERMAT'S EQUATION WITH EXPONENT $2p$

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In [2] Terjanian proved that if p is an odd prime and x, y, z are integers such that $2p$ does not divide either x or y, then $x^{2p}+y^{2p} = z^{2p}$ does not hold.

From the theorem of Terjanian we shall deduce the following

THEOREM. *If p is an odd prime, and x, y, z are integers such that $8p^3$ does not divide x and y, then $x^{2p}+y^{2p} = z^{2p}$ does not hold.*

Proof. We can assume without loss of generality that $(x, y) = (y, z) = (z, x) = 1$, $2|y$. By the theorem of Terjanian we have $2p|y$.

First we prove that $8|y$. From $(z, y) = 1$, $2|y$, it follows that

$$(z^p-y^p, z^p+y^p) = 1.$$

Thus from the equation $x^{2p} = z^{2p}-y^{2p} = (z^p-y^p)(z^p+y^p)$ we obtain

$$(z-y)\frac{z^p-y^p}{z-y} = a^{2p}, \tag{1}$$

where a is an odd positive integer. From $2p|y$ it follows that $p\nmid z-y$, whence

$$\left(z-y, \frac{z^p-y^p}{z-y}\right) = 1. \tag{2}$$

By (1) and (2) we get $(z^p-y^p)/(z-y) = g^2$, where g is an odd positive integer. Consequently,

$$z^{p-1}+z^{p-2}y+z^{p-3}y^2 \equiv 1 \pmod 8.$$

From $2\nmid z$ we get $z^{p-1} \equiv 1 \pmod 8$.

Thus $yz^{p-3}(z+y) \equiv 0 \pmod 8$, and since $2\nmid z$ and $2\nmid z+y$, we have

$$y \equiv 0 \pmod 8. \tag{3}$$

Now we prove that $p^3|y$. From $x^{2p} = (z^p-y^p)(z^p+y^p)$, $(z^p-y^p, z^p+y^p) = 1$ we get $z^p+y^p = t^{2p} = (t^2)^p$, where t is an odd positive integer. By the theorem of Vandiver (see [1], p. 327, theorem 1046), from $x^p+y^p+z^p = 0$, where $(x, y, z) = 1$, $xyz \neq 0$, p is an odd prime, it follows that

$$x^p \equiv x \pmod{p^3}, \qquad y^p \equiv y \pmod{p^3}, \qquad z^p \equiv z \pmod{p^3}.$$

From $z^p+y^p = (t^2)^p$ we deduce $z^p+y^p+(-t^2)^p = 0$, and by the theorem of Vandiver we obtain $y^p \equiv y \pmod{p^3}$. Since $p|y$, $p \geqslant 3$, we have $p^3|y$, and from (3) it follows that $8p^3|y$, which completes the proof of our theorem.

REFERENCES

[1] E. Landau, *Vorlesungen über Zahlentheorie*, New York 1947.
[2] G. Terjanian, *Sur l'équation* $x^{2p}+y^{2p} = z^{2p}$, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris, 285 (1977), p. 973-975.

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