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ON THE HOMEOMORPHIC MEASURE PROPERTY

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Introduction. The classical von Neumann-Oxtoby-Ulam Theorem states the following:

Given non-atomic Borel probability measures μ and λ on I^n such that

(1) $\mu(A) > 0$ *and* $\lambda(A) > 0$ *for all open* $A \subset I^n$,

(2) $\mu(\partial I^n) = \lambda(\partial I^n) = 0$,

there exists a homeomorphism h of I^n onto itself fixing the boundary pointwise such that, for any λ-measurable set S, $\mu(h(S)) = \lambda(S)$.

It is known that the above theorem remains valid if I^n is replaced with any compact finite-dimensional manifold ([3] and [5]) or with I^∞, the Hilbert cube ([6] and [8]). Naturally, in both cases, the condition (2) property in the above theorem remains valid with ∂I^n replaced by ∂X.

Proofs of the Hilbert cube result were done independently at approximately the same time. The approach taken by Oxtoby and Prasad [6], after some essential reductions, is the same constructive method of the original Oxtoby-Ulam Euclidean cube proof. This approach has the virtue that it identifies the type of sets which can remain invariant under the homeomorphism, and thus leads to a Luzin type theorem for approximating Borel measurable transformations of the Hilbert cube onto itself by homeomorphisms. The proof given by Weiss [8] is noteworthy in that it uses the finite-dimensional theorem on fibers over finite-dimensional subspaces of the Hilbert cube. We adopt this method of attack with some necessary refinements to prove that a countable product of finite-dimensional manifolds has the homeomorphic measure property. We remark that even for such a simple example as T^∞, an infinte product of circles, our result does not follow from the I^∞ theorem by any simple identification map.

Homeomorphic measure property for products of compact manifolds. The following device is frequently used to establish the homeomorphic measure property for spaces using the classical theorem for I^n:

Suppose X is a topological space with the property that there exists a continuous function f from I^n onto X which is a homeomorphism of the interior of I^n onto its image. Then if μ and λ are locally positive non-atomic Borel probability measures on X such that $\mu(f(\partial I^n)) = \lambda(f(\partial I^n)) = 0$, then there is a homeomorphism h of X onto itself such that $\mu(h(S)) = \lambda(S)$ for every λ-measurable set S.

This property is shown by applying the von Neumann-Oxtoby-Ulam Theorem on the measures $\mu(f(*))$ and $\lambda(f(*))$ on I^n to find a homeomorphism h' of I^n onto itself such that $\mu(f(h'(*))) = \lambda(f(*))$. (Henceforth, we shall write only $\mu(f \circ h') = \lambda(f)$.) It is then easily shown that $fh'f^{-1}$ is the desired homeomorphism of X onto itself. In the case where X is a compact finite-dimensional manifold, an extension by Brown [1] of a result of Doyle and Hocking [2] establishes the existence of the appropriate function f. By modification techniques which can be found in [3] or [5], for any σ-finite Borel measures μ and λ on X (perhaps with atoms) one can choose f so that $f(\partial I^n)$ is μ- and λ-null. This same extension device is used in the source paper by Oxtoby and Ulam [7] to show that a compact regularly connected finite polyhedron has the homeomorphic measure property.

Notation. Let μ be a Borel measure on a product space $X \times Y$. We denote by μ_X the marginal measures defined by $\mu_X(A) = \mu(A \times Y)$ on X, and by μ_Y those defined by $\mu_Y(B) = \mu_Y(X \times B)$ on Y.

PROPOSITION 1. *Let X be a compact manifold and let Y be a compact regularly connected finite polyhedron. Then $X \times Y$ has the homeomorphic measure property.*

Proof. Let μ and λ be locally positive non-atomic Borel probability measures on $X \times Y$ such that

$$\mu(\partial(X \times Y)) = \mu(X \times \partial Y) = \lambda(X \times \partial Y) = 0.$$

There is a continuous function f of I^m onto X which is a homeomorphism of the interior of I^m onto its image and such that $f(\partial I^m)$ is null for μ_X and λ_X. Let g be a similar map of I^n onto Y with $g(\partial I^n)$ null for μ_Y and λ_Y. Then $f \times g\colon I^{n+m} \to X \times Y$ is a continuous function having the necessary properties to invoke the extension device.

We need a refinement of this idea.

PROPOSITION 2. *Let X be an n-dimensional compact manifold and Y an m-dimensional compact manifold. Suppose μ and λ are locally positive non-atomic Borel probability measures on $X \times Y$ such that μ_X and λ_X are identical. Then there exists a homeomorphism h of $X \times Y$ onto itself satisfying $\mu(h) = \lambda$, which further may be chosen to change the X-coordinate of any point by less than an arbitrary positive amount.*

Proof. Find continuous functions f and g of I^n onto X and I^m onto Y, respectively, which are homeomorphisms restricted to the interiors and such that $f(\partial I^n)$ is μ_X- and λ_X-null and $g(\partial I^m)$ is μ_Y- and λ_Y-null. Now I^n may be subdivided into closed rectangular cells $\{\sigma_j \mid i \leqslant j \leqslant K\}$ so fine that the diameter in the X-metric of $f(\sigma_j)$ is less than the preassigned constant and such that $f(\partial\sigma_j)$ is μ_X- and λ_X-null for each j. Then $f_j \times g$ may be regarded as a continuous map from I^{n+m} onto $f(\sigma_j) \times Y$, which is a homeomorphism of the interior of I^{n+m} onto its range. Furthermore, we have

$$\mu\big(f(\sigma_j) \times Y\big) = \mu_X\big(f(\sigma_j)\big) = \lambda_X\big(f(\sigma_j)\big) = \lambda\big(f(\sigma_j) \times Y\big).$$

Thus there is a homeomorphism h_j of $f(\sigma_j) \times Y$ onto itself such that $\mu(h_j) = \lambda$ for the restrictions of μ and λ. The family of homeomorphisms $\{h_j \mid 1 \leqslant j \leqslant K\}$ is seen to agree on overlap and may be joined to define a single homeomorphism h of $X \times Y$ onto itself with the desired properties.

Let $M = \prod_{i=1}^{\infty} M_i$ be the countably infinite product of finite-dimensional compact manifolds M_i. We assume each manifold M_i has a metric ϱ_i such that the diameter of M_i is 1. The topology on M is defined by the metric ϱ such that, for $\bar{x}, \bar{y} \in M$,

$$\bar{x} = (x_1, x_2, \ldots), \qquad \bar{y} = (y_1, y_2, \ldots),$$

$$\varrho(\bar{x}, \bar{y}) = \sum_{i=1}^{\infty} 2^{-i} \varrho_i(x_i, y_i).$$

On each finite product of the form $M_1 \times \ldots \times M_N$ we define the metric σ_N by the formula

$$\sigma_N\big((x_1, \ldots, x_N), (y_1, \ldots, y_N)\big) = \max_{1 \leqslant i \leqslant N} \varrho_i(x_i, y_i).$$

We let π_N denote the projection map of M onto $M_1 \times \ldots \times M_N$.

Proposition 3. *Let μ and λ be locally positive non-atomic Borel probability measures on M. There exists a homeomorphism of M onto itself such that the measures on $M_1 \times \ldots \times M_N$ defined by $\mu(h \circ \pi_N^{-1})$ and $\lambda(h \circ \pi_N^{-1})$ are non-atomic for all N. In fact, the family of homeomorphisms with this property is a dense G_δ-set in the space of all homeomorphisms of M onto itself with the topology of uniform convergence.*

Proof. Evidently, it suffices to show this for $N = 1$. Let $\mathscr{H}$ be the space of homeomorphisms of M onto itself with the topology of uniform convergence. Then $\mathscr{H}$ is completely metrizable, thus Baire. Put

$$A_K = \{h \in \mathscr{H} \mid (\exists x \in M_1)\ \mu\big(h \circ \pi_1^{-1}(x)\big) \geqslant 1/K\}.$$

We show A_K is closed nowhere dense. It follows immediately that any h in $\mathscr{G}_\mu = \mathscr{H} - \bigcap_{K=1}^{\infty} A_K$ is such that $\mu(h \circ \pi_1^{-1})$ is non-atomic. Taking $\mathscr{G}_\mu \cap \mathscr{G}_\lambda$, we produce the required dense G_δ-set.

By a standard argument, A_k is closed. Let $h \in A_K$ and $\varepsilon > 0$ be given. For sufficiently large N, each atom of $\mu(h \circ \pi_N^{-1})$ has weight less than $1/2K$. Write $\mu(h \circ \pi_N^{-1})$ as $\mu_1 + \mu_2$, where μ_1 is purely atomic and μ_2 is non-atomic. We may choose a finite subset S of the support of μ_1 such that

$$\mu_1(M_1 \times \ldots \times M_N) - \mu_1(S) < 1/2K.$$

Let δ be such that $\varrho(x, y) < \delta$ implies $\varrho\big(h(x), h(y)\big) < \varepsilon$. It is readily seen that there exists a homeomorphism g of $M_1 \times \ldots \times M_N$ onto itself which moves no point as much as δ and which moves the points of S so that the projection of $g(S)$ onto M_1 is one-to-one.

Let α be any locally positive non-atomic Borel probability measure on $M_1 \times \ldots \times M_N$ such that α_{M_1} is non-atomic. One can find a homeomorphism f of $M_1 \times \ldots \times M_N$ onto itself as close to the identity homeomorphism as desired such that $\mu_2(g \circ f)$ is equivalent to α. (Thus $\mu_2(g \circ f)_{M_1}$ is non-atomic.) This follows easily from the short argument in [4] and is found in [3]. Take f so close to the identity that $g \circ f$ moves no point as much as δ and that the projection of $g \circ f(s)$ onto M_1 remains one-to-one. Extending $g \circ f$ to a homeomorphism h of M by

$$h' = (g \circ f) \times \Big(\text{identity on } \prod_{i=N+1}^{\infty} M_i\Big),$$

we infer that $h \circ h'$ does not belong to A_K but is within an ε-neighborhood of h. Thus A_K is nowhere dense and the proof is complete.

THEOREM. *A countable product of finite-dimensional compact manifolds has the homeomorphic measure property.*

Proof. A finite product of finite-dimensional compact manifolds is a finite-dimensional compact manifold. It suffices to show M has the homeomorphic measure property.

Let μ and λ be locally positive non-atomic Borel probability measures on M. By Proposition 3, we may assume $\mu(\pi_n^{-1})$ and $\lambda(\pi_n^{-1})$ are non-atomic for each positive integer n by their equivalence to measures which have this property. Applying the von Neumann-Oxtoby-Ulam Theorem to $\mu(\pi_1^{-1})$ and $\lambda(\pi_1^{-1})$, we find a homeomorphism h_1 of M_1 onto itself such that $\mu(\pi_1^{-1} \circ h_1) = \lambda(\pi_1^{-1})$.

Note that if g is any homeomorphism of $M_1 \times \ldots \times M_N$ onto itself, then g may be naturally extended to a homeomorphism $\tilde{g}$ of M onto itself by

$$\tilde{g} = g \times \Big(\text{identity on } \prod_{i=N+1}^{\infty} M_i\Big).$$

Evidently, $\pi_N^{-1} \circ g = \tilde{g} \circ \pi_N^{-1}$. Applying this to the above, we have

$$\mu(\pi_1^{-1} \circ h) = \mu(\tilde{h}_1 \circ \pi_1^{-1}) = \lambda(\pi_1^{-1}).$$

Let $\mu_1 = \mu(\tilde{h}_1)$.

Now, $\mu_1(\pi_2^{-1})$ and $\lambda(\pi_2^{-1})$ satisfy the hypotheses of Proposition 2 on $M_1 \times M_2$, so there exists a homeomorphism h_2 of $M_1 \times M_2$ onto itself such that

(1) $\mu_1(\pi_2^{-1} \circ h_2) = \mu_1(\tilde{h}_2 \circ \pi_2^{-1}) = \lambda(\pi_2^{-1})$,

(2) $\sigma_1(\pi_1 \circ \tilde{h}_1 \circ \tilde{h}_2(\bar{x}),\ \pi_1 \circ \tilde{h}_1(\bar{x})) < 1/4$ for every $\bar{x} \in M$,

(3) $\varrho(\tilde{h}_2(\bar{x}), \bar{x}) < 1/2$ for every $\bar{x} \in M$.

Conditions (2) and (3) can be satisfied by choosing h_2 which changes the M_1-coordinate by no more than an amount determined by $\tilde{h}_1$. Let $\mu_2 = \mu_1(\tilde{h}_2)$.

Continuing inductively, suppose $\mu_1, \ldots, \mu_n$; $\tilde{h}_1, \ldots, \tilde{h}_n$ have been defined so that $\mu_n(\pi_{n+1}^{-1})$ and $\lambda(\pi_{n+1}^{-1})$ satisfy the hypotheses of Proposition 2 on $(M_1 \times \ldots \times M_n) \times M_{n+1}$. Thus there exists a homeomorphism h_{n+1} of $M_1 \times \ldots \times M_{n+1}$ onto itself such that

(1) $\mu_n(\pi_{n+1}^{-1} \circ h_{n+1}) = \mu_n(\tilde{h}_{n+1} \circ \pi_{n+1}^{-1}) = \lambda(\pi_{n+1}^{-1})$,

(2) $\sigma_n(\pi_n \circ \tilde{h}_1 \circ \tilde{h}_2 \circ \ldots \circ \tilde{h}_{n+1}(\bar{x}),\ \pi_n \circ \tilde{h}_1 \circ \ldots \circ \tilde{h}_n(\bar{x})) < 1/2^{n+1}$ for every $\bar{x} \in M$,

(3) $\varrho(\tilde{h}_{n+1}(\bar{x}), \bar{x}) < 1/2^n$ for every $\bar{x} \in M$.

Let $\mu_{n+1} = \mu_n(\tilde{h}_{n+1})$.

Now, $\tilde{h}_1 \circ \tilde{h}_2 \circ \ldots \circ \tilde{h}_n$ converges to a homeomorphism h because (2) implies that the sequence converges to a continuous function and it follows from (3) that $\tilde{h}_n^{-1} \circ \tilde{h}_{n-1}^{-1} \circ \ldots \circ \tilde{h}_1^{-1}$ converges to a continuous function. Consequently, $\tilde{h}_1 \circ \tilde{h}_2 \circ \ldots \circ \tilde{h}_n$ is Cauchy in a complete metric on $\mathscr{H}$. Clearly, h has the required property $\mu(h) = \lambda$.

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