COLLOQUIUM MATHEMATICUM

VOL. LVII 1989 FASC. 1

ON MANIFOLDS SATISFYING SOME CURVATURE CONDITIONS

BY

RYSZARD DESZCZ AND $\boxed{\text{WIESŁAW GRYCAK}}$ (WROCŁAW)

1. Introduction. Let M be a connected n-dimensional ($n \geqslant 4$) Riemannian manifold of class C^∞ with not necessarily definite metric g. A manifold M is said to be *pseudo-symmetric* [6] if the condition

$$R \cdot R = LQ(g, R) \tag{1}$$

holds on M, where R is the curvature tensor of M, L a function on M, and the tensors $R \cdot R$ and $Q(g, R)$ are defined as in (4) and (5), respectively. It is clear that any semi-symmetric manifold ($R \cdot R = 0$; cf. [10]) is pseudo-symmetric. Examples of non semi-symmetric pseudo-symmetric manifolds are given in [2], [3] and [6]. These manifolds may or may not be conformally flat. The condition (1) arose in the study of totally umbilical submanifolds of semi-symmetric manifolds [1] and in considering geodesic mappings [11] (see also [7] and [8]).

It is easy to verify that any pseudo-symmetric manifold satisfies the condition

$$R \cdot C = LQ(g, C), \tag{2}$$

where C is the Weyl conformal curvature tensor of M, and the tensor $Q(g, C)$ is defined as in (5). The converse statement fails in general. An example of conformally flat and non pseudo-symmetric manifold is described in [2]. Furthermore, in [4] (Theorem 1) an example of a four-dimensional non-conformally flat and non pseudo-symmetric manifold satisfying (2) is given. In the present paper we prove that if $\dim M \geqslant 5$, then (2) implies (1) at every point of M at which $C \neq 0$. From this it follows immediately that every analytic and non-conformally flat manifold M of dimension $\geqslant 5$ and satisfying (2) is necessarily pseudo-symmetric.

2. Preliminaries. Let M be an n-dimensional ($n \geqslant 4$) Riemannian manifold with not necessarily definite metric g. We denote by g_{ij}, R_{hijk}, S_{ij} and

$$(3)\qquad C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{hk}S_{ij} - g_{hj}S_{ik} + g_{ij}S_{hk} - g_{ik}S_{hj}) + \frac{K}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{hj}g_{ik})$$

and K the local components of the metric g, the Riemann-Christoffel curvature tensor R, the Ricci tensor S, the Weyl conformal curvature tensor C and the scalar curvature K of M, respectively. For a tensor B of type $(0, 4)$, with local components B_{hijk}, we define tensors $B(1)$, $R\cdot B$ and $Q(g, B)$ (cf. [6]) by the formulas

$$B(1)_{hijk} = \frac{K(B)}{n(n-1)}(g_{hk}g_{ij} - g_{hj}g_{ik}),$$

$$(4)\qquad (R\cdot B)_{hijklm} = -B_{sijk}R^{s}{}_{hlm} - B_{hsjk}R^{s}{}_{ilm} - B_{hisk}R^{s}{}_{jlm} - B_{hijs}R^{s}{}_{klm},$$

$$(5)\qquad Q(g, B)_{hijklm} = g_{hl}B_{mijk} + g_{il}B_{hmjk} + g_{jl}B_{himk} + g_{kl}B_{hijm} - g_{hm}B_{lijk} - g_{im}B_{hljk} - g_{jm}B_{hilk} - g_{km}B_{hijl},$$

where $K(B) = g^{hk}g^{ij}B_{hijk}$. Similarly as the tensors $R\cdot B$ and $Q(g, B)$ we can define the tensors $R\cdot A$ and $Q(g, A)$, where A is a tensor of type $(0, 2)$. By a *generalized curvature tensor* we mean a tensor B satisfying $B_{hijk} = -B_{hikj} = B_{jkhi}$ and $B_{hijk} + B_{hjki} + B_{hkij} = 0$ (the first Bianchi identity).

In the next section we shall need the following lemma:

LEMMA 1 ([8], Lemma 2). *If B is a generalized curvature tensor at a point x of a Riemannian manifold M such that $R\cdot B = \alpha Q(g, B)$, and if A and D are symmetric tensors of type $(0, 2)$ at x satisfying the condition $R\cdot A = Q(g, D)$, then*

$$\left(E - \frac{1}{n}\operatorname{tr}(E)g\right)(B - B(1)) = 0,$$

where $E = D - \alpha A$, and α is a number.

3. Main results.

THEOREM 1. *Let M be a Riemannian manifold of dimension $n \geqslant 5$ satisfying the condition $R\cdot C = LQ(g, C)$ and let U be the open subset of M on which $C \neq 0$. Then on U the condition $R\cdot R = LQ(g, R)$ holds true.*

Proof. Symmetrizing the equality

$$(R\cdot C)_{hijklm} = LQ(g, C)_{hijklm}$$

with respect to the pairs (h, i), (j, k) and (l, m) and applying the relations

$$Q(g, C)_{hijklm} + Q(g, C)_{jklmhi} + Q(g, C)_{lmhijk} = 0$$

([2], Lemma 1.1 (i)), (3) and

$$(R\cdot R)_{hijklm}+(R\cdot R)_{jklmhi}+(R\cdot R)_{lmhijk} = 0$$

([12], equality (26), p. 64), we obtain

$$\begin{aligned} g_{ij}(R\cdot S)_{hklm}-g_{ik}(R\cdot S)_{hjlm}+g_{hk}(R\cdot S)_{ijlm}-g_{hj}(R\cdot S)_{iklm} \\ +g_{kl}(R\cdot S)_{jmhi}-g_{km}(R\cdot S)_{jlhi}+g_{jm}(R\cdot S)_{klhi}-g_{jl}(R\cdot S)_{kmhi} \\ +g_{hm}(R\cdot S)_{lijk}-g_{mi}(R\cdot S)_{lhjk}+g_{li}(R\cdot S)_{mhjk}-g_{lh}(R\cdot S)_{mijk} = 0. \end{aligned}$$

Contracting the above equation with g^{ij} we find

$$\begin{aligned} (6)\qquad (n-2)(R\cdot S)_{hklm}+g_{kl}V_{hm}-g_{km}V_{lh}+g_{lh}V_{km}-g_{hm}V_{kl} \\ +(R\cdot S)_{lkhm}+(R\cdot S)_{hmlk}-(R\cdot S)_{mkhl}-(R\cdot S)_{hlmk} = 0, \end{aligned}$$

where V is the symmetric tensor of type (0, 2) of local components

$$V_{kl} = g^{hj}(R\cdot S)_{hklj}.$$

This is just as (12) in [9]. From (6), in the same way as in the proof of Proposition 1 in [9], on U we obtain

$$(n-4)\left(R\cdot S-Q\left(g,\ -\frac{1}{n}V\right)\right) = 0,$$

whence

$$R\cdot S = Q\left(g,\ -\frac{1}{n}V\right). \tag{7}$$

This relation corresponds to (13) in [9]. Now the relation (7), in virtue of Lemma 1 and the assumption $C \neq 0$, gives $R\cdot S = LQ(g, S)$. Substituting this into (2) we get easily (1), which completes the proof.

As an immediate consequence of Theorem 1 we obtain the following corollary:

Corollary 1. *If M, $\dim M \geqslant 5$, is an analytic Riemannian manifold satisfying $R\cdot C = LQ(g, C)$, then M is conformally flat or pseudo-symmetric.*

The last result, in the case where $L = 0$ on M, is proved in [9].

Since the Weyl conformal curvature tensor of every totally umbilical submanifold N of a manifold M satisfying the condition (2) also satisfies the condition of this form ([5], Lemma 2), Corollary 1 yields

Corollary 2. *If N, $\dim N \geqslant 5$, is an analytic and connected totally umbilical submanifold of an analytic Riemannian manifold M satisfying the condition $R\cdot C = LQ(g, C)$, then N is conformally flat or pseudo-symmetric.*

REFERENCES

[1] A. Adamów and R. Deszcz, *On totally umbilical submanifolds of some class of Riemannian manifolds*, Demonstratio Math. 16 (1983), pp. 39–59.

[2] J. Deprez, R. Deszcz and L. Verstraelen, *Examples of pseudo-symmetric conformally flat warped products*, Chinese J. Math., in print.

[3] — *Pseudo-symmetry conditions on hypersurfaces of Euclidean spaces and on Kählerian manifolds*, Ann. Fac. Sci. Toulouse Math. (5), in print.

[4] A. Derdziński, *Exemples de métriques de Kaehler et d'Einstein autoduales sur le plan complexe, Géométrie riemannienne en dimension* 4 (Séminaire Arthur Besse 1978–1979), Cedic–Fernand Nathan, Paris 1981, pp. 334–346.

[5] R. Deszcz, S. Ewert-Krzemieniewski and J. Policht, *On totally umbilical submanifolds of conformally birecurrent manifolds*, Colloq. Math. 55 (1988), pp. 79–96.

[6] R. Deszcz and W. Grycak, *On some class of warped product manifolds*, Bull. Inst. Math. Acad. Sinica 15 (1987), pp. 311–322.

[7] R. Deszcz and M. Hotloś, *On geodesic mappings in pseudo-symmetric manifolds*, ibidem 16 (1988), pp. 251–262.

[8] — *Notes on pseudo-symmetric manifolds admitting special geodesic mappings*, Soochow J. Math., in print.

[9] W. Grycak, *Riemannian manifolds with a symmetry condition imposed on the 2-end derivative of the conformal curvature tensor*, Tensor, N. S., to appear.

[10] Z. I. Szabó, *Structure theorems on Riemannian space satisfying $R(X, Y)\cdot R = 0$, I. The local version*, J. Differential Geom. 17 (1982), pp. 531–582.

[11] P. Venzi, *On geodesic mappings in Riemannian or pseudo-Riemannian manifolds*, Tensor, N. S., 32 (1978), pp. 193–198.

[12] A. G. Walker, *On Ruse's spaces of recurrent curvature*, Proc. London Math. Soc. 50 (1950), pp. 36–64.

DEPARTMENT OF MATHEMATICS
AGRICULTURAL ACADEMY
UL. C. NORWIDA 25
50-375 WROCŁAW, POLAND

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY
WYBRZEŻE WYSPIAŃSKIEGO 27
50-376 WROCŁAW, POLAND

Reçu par la Rédaction le 29.4.1987