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NOTE ON A REPRESENTATION OF UNIVERSAL ALGEBRAS AS SUBDIRECT POWERS

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M. I. Gould and G. Grätzer have proved a general theorem concerning the problem mentioned in the title. The original proof of their theorem seems to be rather complicated. I have succeeded in proving this theorem by other means, not using the notion of an inverse limit. I believe that my proof is simpler than the original one and I hope that it will stimulate further study in representations of algebras.

First let us recall some definitions. They can be found in Grätzer's book [2].

Let $\mathscr{A} = \langle A; F\rangle$ be a universal algebra. An algebra is called a *subdirect power of* $\mathscr{A}$ if it is isomorphic to a suitable subalgebra $\mathscr{B} = \langle B; F\rangle$, $B \subset A^I$ (where I is a suitable set), of a direct power $\mathscr{A}^I$ and if

$$Be_i^I = A \text{ for each projection } e_i^I, \quad i \in I.$$

(The projection is a mapping $e_i^I\colon A^I \to A$, $i \in I$, such that

$$e_i^I(\{x_j\}_{j\in I}) = x_i$$

is fulfilled for each family $\{x_j\}_{j\in I} \in A^I$.)

An *algebraic function in* $\mathscr{A}$ is a mapping obtained from a polynomial by substituting fixed elements for certain variables. An *algebraic identity of* $\mathscr{A}$ is a statement of the form

$$f(x_1, \ldots, x_k) = g(x_1, \ldots, x_k) \quad \text{for each } x_1, \ldots, x_k \in A,$$

where f and g are algebraic functions and k a non-negative integer.

The algebra $\mathscr{A}$ is *algebraically complete* if, for any non-negative integer n, each mapping $f\colon A^n \to A$ is an algebraic function in $\mathscr{A}$.

Now let us give the new proof of the general theorem, which was proved by M. I. Gould and G. Grätzer in 1967 [1] and which states:

Let $\mathscr{B} = \langle B; F\rangle$ *be a universal algebra containing a subalgebra* $\mathscr{A} = \langle A; F\rangle$ *such that* $1 < \operatorname{card} A < \infty$,

(A) *$\mathscr{A}$ is algebraically complete, and*

(B) *each algebraic identity of $\mathscr{A}$ is an algebraic identity of $\mathscr{B}$.*

Then $\mathscr{B}$ is a normal subdirect power (1) *of $\mathscr{A}$.*

The original proof of the normality of $\mathscr{B}$ is not complicated. It can hardly be simplified and we will not deal with it at all.

Remark. If $f\colon A^k \to A$ is a mapping, where k is a non-negative integer, and the condition (A) is fulfilled, then there is an algebraic function g in $\mathscr{B}$ such that

$$f(x_1, \ldots, x_k) = g(x_1, \ldots, x_k) \quad \text{for each } x_1, \ldots, x_k \in A.$$

If such a function g is chosen, we shall call it *extension* of f. We shall identify f and g, and both f and g will be denoted by the same symbol f.

LEMMA. *Let the suppositions of the theorem of Gould and Grätzer be fulfilled and elements $\xi, \eta \in B$, $\xi \neq \eta$, be given. Then there is an endomorphism ε of $\mathscr{B}$ such that*

(1) *$a\varepsilon = a$ for all $a \in A$, and*

(2) *$\xi\varepsilon \neq \eta\varepsilon$.*

Proof. Let n denote the cardinal number of the set A, $m = n-1$. We identify both sets, $A = n = \{0, 1, \ldots, m\}$; consequently, A is an ordered set and a distributive lattice $\langle A; \vee, \wedge\rangle$, where $a \vee b = \text{l.u.b.}(\{a, b\})$ and $a \wedge b = \text{g.l.b.}(\{a, b\})$ for $a, b \in A$. By the Remark, the operations $\vee$ and $\wedge$ can be extended onto B^2. The axioms of a distributive lattice (and the relations $0 \subset a \subset m$) can be written in the form of algebraic identities of $\mathscr{A}$ and, by (B), they remain true in $\mathscr{B}$. Thus $\mathscr{B} = \langle B; \vee, \wedge\rangle$ is a distributive lattice, 0 is the least element, and m is the greatest one.

Let us recall that a prime ideal P is characterized by the properties:

(x) $x, y \in P \Rightarrow x \vee y \in P$,

(y) $x \vee y = y, y \in P \Rightarrow x \in P$,

(z) $\wedge \{x_i; i \in k\} \in P \Rightarrow$ there is $i \in k$ such that $x_i \in P$, where k is a positive integer.

We define for $\alpha, a \in A$:

$$E_a^\alpha = \begin{cases} 0 & \text{if } a = \alpha, \\ m & \text{if } a \neq \alpha. \end{cases}$$

We may suppose, by the Remark, that, for each $\alpha \in A$, the mapping E_a^α is defined for each $a \in B$ and it is an algebraic function in $\mathscr{B}$.

(1) For the definition of a normal subdirect power, see [2], p. 150.

The following relations (a), (b) and (c) are evidently algebraic identities of $\mathscr{A}$:

(a) $a = \bigwedge_{\alpha \in A} (a \vee E_a^\alpha)$,

(b) $0 = \bigwedge_{\alpha \in A} E_a^\alpha$,

(c) for each $a, b \in A$

$$E_a^\alpha \vee E_b^\alpha = \begin{cases} E_a^\alpha & \text{if } a = b, \\ m & \text{if } a \neq b. \end{cases}$$

Therefore, by (B), they are also algebraic identities of $\mathscr{B}$.

There is an element $b \in A$ such that either $E_b^\xi \notin (E_b^\eta]$ or $E_b^\eta \notin (E_b^\xi]$; otherwise $E_a^\xi = E_a^\eta$ for each $a \in A$ and $\xi = \eta$, by (a). (The symbol $(E]$ denotes the principal ideal $\{x;\ x \vee E = E\}$.) Suppose the first case is true (the second would be treated analogically). By the theorem of Birkhoff and Stone [2], there is a prime ideal P of $\langle B;\ \vee, \wedge \rangle$ such that

(3) $E_b^\eta \in (E_b^\eta] \subset P$ and

(4) $E_b^\xi \notin P$.

Let $\varepsilon\colon B \to A$ be a mapping such that

(5) $\alpha\varepsilon = a \Leftrightarrow E_a^\alpha \in P$.

It is easy to see that

1) ε is well-defined (if $\alpha\varepsilon = a$ and $\alpha\varepsilon = c$, $a \neq c$, then, by (c), (5) and (x), $m = E_a^\alpha \vee E_c^\alpha \in P$, which is not possible, in view of (4) and (y)),

2) ε is defined for each $\alpha \in B$ (see (b), (y), (3) and (z)),

3) $\xi\varepsilon \neq \eta\varepsilon$ (see (3), (4) and (5)),

4) $a\varepsilon = a$ for $a \in A$ (because $E_a^a = 0 \in P$),

5) ε is a homomorphism.

Let a k-ary operation $f \in F$ and elements $\alpha_1, \ldots, \alpha_k \in B$ be given. The relation

$$E_{f(\alpha_1\varepsilon, \ldots, \alpha_k\varepsilon)}^{f(\beta_1, \ldots, \beta_k)} \vee \left(\bigvee_{i=1}^{k} E_{\alpha_i\varepsilon}^{\beta_i}\right) = \bigvee_{i=1}^{k} E_{\alpha_i\varepsilon}^{\beta_i}$$

is fulfilled for any $\beta_1, \ldots, \beta_k \in A$. (Both sides equal 0 or m according to whether $\beta_i = \alpha_i\varepsilon$ for all i or not for all i.)

Therefore this relation is an identity of $\mathscr{A}$ and, by (B), of $\mathscr{B}$ as well. If $\beta_i = \alpha_i$ for each $i = 1, \ldots, k$, then the element on the right-hand side belongs to P (by (5) and (x)); hence the element $E_{f(\alpha_1\varepsilon, \ldots, \alpha_k\varepsilon)}^{f(\beta_1, \ldots, \beta_k)}$ belongs to P. By (5) we get

$$f(\alpha_1, \ldots, \alpha_k)\varepsilon = f(\alpha_1\varepsilon, \ldots, \alpha_k\varepsilon).$$

Thus the proof of the lemma is finished.

Now it is easy to prove the theorem of Gould and Grätzer. Let $\{\varepsilon_i; i \in I\}$ be the set of all endomorphisms $\varepsilon_i: \mathscr{B} \to \mathscr{A}$ fulfilling condition (1) and let the mapping $\varphi: B \to A^I$ be defined by

$$x\varphi = \{x\varepsilon_i\}_{i \in I} \in A^I \quad \text{for each } x \in B.$$

Evidently, φ is a homomorphism and, by the Lemma, it is injective. Therefore φ is an isomorphism. Condition (1) implies that $\mathscr{B}$ is a subdirect power of $\mathscr{A}$. As we have already said, we will not prove the normality of this representation.

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REFERENCES

[1] M. I. Gould and G. Grätzer, ***Boolean extensions and normal subdirect power of finite universal algebras,*** Mathematische Zeitschrift 99 (1967), p. 16-25.

[2] G. Grätzer, *Universal algebra,* Van Nostrand 1968.

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