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POINT PARTITION NUMBERS AND GENERALIZED NORDHAUS–GADDUM PROBLEMS

BY

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1. Introduction. The graphs under consideration here are *simple graphs*, i.e. finite undirected graphs with neither loops nor multiple lines. The point set of a graph G is denoted by $V(G)$, while the line set is denoted by $E(G)$. We denote by $\delta(G)$ and $\Delta(G)$ the minimum degree and the maximum degree of a graph G, respectively. The subgraph induced by a set U of points of G, denoted by $\langle U\rangle$, has U as its point set and contains all those lines of G which are incident to two points of U. In [10], a graph G was defined to be *k-degenerate* for a nonnegative integer k if $\delta(H) \leqslant k$ for each induced subgraph H of G. It is clear that a graph is totally disconnected if and only if it is 0-degenerate. *Forests* (or *acyclic graphs*) are precisely the 1-degenerate graphs. According to [11], the *point partition number* $\varrho_k(G)$ of a graph G, for a given nonnegative integer k, is the smallest number of sets into which the point set $V(G)$ can be partitioned so that each set induces a k-degenerate subgraph of G. It is easy to see that the parameters $\varrho_0(G)$ and $\varrho_1(G)$ are the *chromatic number* and the *point arboricity* of a graph G, respectively. A graph G is said to be *p-critical* with respect to ϱ_k if $\varrho_k(G) = p$, but $\varrho_k(G-v) = p-1$ for each point v of G. The definitions not given here may be found in [6].

2. The point partition number. Lick [11] has proved the following

PROPOSITION 1. *If a graph G is p-critical with respect to ϱ_k, where k is a nonnegative integer, then*

$$\delta(G) \geqslant (k+1)(p-1).$$

For any real number r, we use the symbols $[r]$ and $\{r\}$ to denote the greatest integer not exceeding r and the least integer not less than r, respectively.

THEOREM 1. *Let f be a real-valued function on graphs with the properties*
(a) *$f(H) \leqslant f(G)$ for each subgraph H of G,*
(b) *$f(G) \geqslant \delta(G)$ for each graph G.*

Then

$$\varrho_k(G) \leqslant \left[\frac{f(G)}{k+1}\right] + 1.$$

Proof. Let $\varrho_k(G) = p+1$ and let H be a subgraph of G, $(p+1)$-critical with respect to ϱ_k. Proposition 1 implies $\delta(H) \geqslant (k+1)p$, and (a) yields that $f(H) \leqslant f(G)$. Thus, by (b), we have

$$p(k+1) \leqslant \delta(H) \leqslant f(H) \leqslant f(G).$$

Hence

$$\varrho_k(G) = p+1 \leqslant \left[\frac{f(G)}{k+1}\right] + 1.$$

For $k = 0$ this theorem gives the result of Szekeres and Wilf [16].

Let G be an m-degenerate graph and $m > k \geqslant 0$. Denote by $M_k(G)$ the maximum number of points of G which induce a k-degenerate subgraph of G. Let $T_k(G)$ denote the minimal cardinality of a set $T \subset V(G)$ such that for every non-k-degenerate subgraph H of G we have $V(H) \cap T \neq \emptyset$.

It is easy to see that $M_0(G)$ and $T_0(G)$ are the point independence number and the point covering number of G, respectively. If a graph G is k-degenerate, then every subgraph of G is k-degenerate. Hence this property is hereditary. By the theorem of Hedetniemi [7] we obtain a generalization of the theorem of Gallai [4]:

THEOREM 2. *For an m-degenerate* $(m > k \geqslant 0)$ *graph* G,

$$M_k(G) + T_k(G) = |V(G)|.$$

THEOREM 3. *For an m-degenerate* $(m > k \geqslant 0)$ *graph* G,

$$\delta(G) \leqslant T_k(G) + k.$$

Proof. Let $N \subset V(G)$ be a set such that for every non-k-degenerate subgraph H of G we have $V(H) \cap N \neq \emptyset$ and $|N| = T_k(G)$.

Let $G' = \langle V(G) \backslash N \rangle$ and assume that $\delta(G') = d_{G'}(v')$, $v' \in V(G) \backslash N$. Obviously, G' is k-degenerate. Hence and from the definition of the k-degenerate graphs we obtain $d_{G'}(v') \leqslant k$. Thus

$$d_G(v') \leqslant k + |N| = k + T_k(G). \tag{$*$}$$

Since $\delta(G) \leqslant d_G(v')$, by $(*)$ the theorem is proved.

COROLLARY 1 ([9], Proposition 7). *Let G be an m-degenerate* $(m > k \geqslant 0)$ *graph with n points. Then*

$$\left\{\frac{n}{M_k(G)}\right\} \leqslant \varrho_k(G) \leqslant \left\{\frac{n - M_k(G)}{k+1}\right\} + 1.$$

Proof. The lower bound is obvious.

By Theorem 3 and the definition of $T_k(G)$ it is easy to see that the function $f(G) = T_k(G)+k$ satisfies the conditions (a) and (b) of Theorem 1. Thus

$$(**) \qquad \varrho_k(G) \leqslant \left[\frac{T_k(G)+k}{k+1}\right]+1$$

and, by Theorem 2, (**), and elementary calculations, we obtain the upper bound.

Now, we review some functions which satisfy the conditions (a) and (b) of Theorem 1.

(i) The minimum number of lines in any cutset of the connected graph G is called the *line-connectivity* of G and is denoted by $l(G)$.

In [12] Matula defined the *strength* $s(G)$ of a graph G as follows:

$$s(G) = \max\{l(H)\colon H \text{ is a subgraph of } G\}.$$

It is obvious that $s(G)$ satisfies the conditions (a) and (b).

(ii) (See [16].) For a graph G of order n, let $N(G)$ denote the $n\times n$ adjacency matrix of G. Let $\lambda = \lambda(G)$ be the largest eigenvalue of $N(G)$.

(iii) $\Delta(G) = \max\limits_{v\in V(G)} d(v)$.

(iv) (See [17].) If $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $d(v_1) \geqslant d(v_2) \geqslant \ldots \geqslant d(v_n)$, then we put

$$g(G) = \max_{1\leqslant i\leqslant n} \min\{i, d(v_i)+1\}.$$

Of course, the functions $\lambda(G)$, $\Delta(G)$, and $g(G)$ satisfy the conditions (a) and (b), and $g(G) \leqslant \Delta(G)$ (see, e.g., [13]). From Theorem 1 and (i)–(iv) we obtain the four upper bounds for $\varrho_k(G)$.

The bound for chromatic number (i.e., for $k = 0$) given in the term of $\lambda(G)$ was proved by Wilf in [18]. Moreover, Szekeres and Wilf in [16] proved that the smallest function f which satisfies the conditions (a) and (b) of Theorem 1 is

$$f(G) = \max_{G'\subset G} \min_{v\in V(G')} d_{G'}(v).$$

LEMMA 1. *For a k-degenerate graph G,*

$$\varrho_{k-r}(G) \leqslant r+1 \quad \textit{for } 0 \leqslant r \leqslant k.$$

Proof. Obviously, for $r = 0$ the lemma is true.

Let $k \geqslant r \geqslant 1$ and suppose that there exists a graph G such that $\varrho_{k-r}(G) > r+1$. Therefore, Theorem 1 and (i) imply

$$r+1 < \varrho_{k-r}(G) \leqslant 1+\frac{s(G)}{k-r+1}.$$

Since $s(G) \leqslant k$, we have $r(k-r+1) < k$, and by elementary calculations we obtain $(k-r)(r-1) < 0$, in contradiction with the nonnegativity of values of both factors.

THEOREM 4. *If* $\varrho_k(G) = t$, *then* $\varrho_{k-r}(G) \leqslant t(r+1)$, $k \geqslant r \geqslant 0$.

Proof. Since $\varrho_k(G) = t$, we have the partition $V(G) = V_1 \cup \ldots \cup V_t$, where each V_i induces a k-degenerate subgraph. The application of Lemma 1 to each $\langle V_i \rangle$ completes the proof.

COROLLARY 2. *We have* $(r+1)^{-1} \varrho_{k-r}(G) \leqslant \varrho_k(G)$.

3. Some generalizations of Nordhaus–Gaddum type theorems. In [14] Nordhaus and Gaddum showed that for any graph G of order n the following inequalities hold:

(i) $$\{2\sqrt{n}\} \leqslant \varrho_0(G) + \varrho_0(\bar{G}) \leqslant n+1,$$

(ii) $$n \leqslant \varrho_0(G)\varrho_0(\bar{G}) \leqslant \left[\left(\frac{n+1}{2}\right)^2\right].$$

Furthermore, these bounds are the best possible for infinitely many values of n. Finck [3] characterized all graphs G such that equality holds in any of the four inequalities given in (i) and (ii).

The determination of upper and lower bounds (preferably sharp bounds) for $f(G)+f(\bar{G})$ and $f(G)\,f(\bar{G})$, where G is a graph of order n, is called a *Nordhaus–Gaddum problem* for a given graph-theoretic parameter f and a positive integer n.

A survey of some results of this type is given in [1]. There are several variations and generalizations of this problem. For example, one might consider distinct but related parameters f_1 and f_2 and develop bounds for $f_1(G)+f_2(\bar{G})$ and $f_1(G)\,f_2(\bar{G})$.

In 1969, Gupta [5] considered the problems of this type for the chromatic, achromatic, and pseudochromatic numbers. Analogous results for hypergraphs are obtained in [13] and [8].

In addition, for a parameter f and graphs G_1 and G_2 related in some prescribed manner, the problem exists to investigate bounds for $f(G_1)+f(G_2)$ and $f(G_1)\,f(G_2)$.

In 1964 Dirac [2] considered a problem of this type for the chromatic number, and in 1974 Schürger [15] considered such problems for the chromatic number and the point independence number.

In the sequel we consider theorems of the last type for graphs where we take $M_k(G)$ as the parameter f and the point partition number $\varrho_k(G)$.

In this part of the paper we consider the graphs G and H on the same set of points, i.e., $V(G) = V(H) = V$. By the *union* $G \cup H$ of such graphs we mean the graph which has the point set $V(G) \cup V(H) = V$ and the line set $E(G) \cup E(H)$. Similarly, $G' = G \cap H$ has $V(G') = V(G) \cap V(H) = V$ and $E(G') = E(G) \cap E(H)$.

We begin with the following

LEMMA 2. $\varrho_k(G \cup H) \leqslant (k+1)\varrho_k(G)\varrho_k(H)$.

Proof. First, let $\varrho_k(G) = t$ and $\varrho_k(H) = s$. Then we have a partition $V = V_1 \cup \ldots \cup V_t$, where each $\langle V_i \rangle$ is k-degenerate in G. According to Lemma 1, for $r = k$ there exists a partition

$$V_i = V_{i,1} \cup \ldots \cup V_{i,k+1}, \quad i = 1, \ldots, t,$$

where each set $V_{i,j}$ is independent.

Now, let V be partitioned into s sets (i.e., $V = U_1 \cup \ldots \cup U_s$) such that each $\langle U_j \rangle$ is k-degenerate in H. It is easy to see that each set $V_{i,l} \cap U_j$ induces a k-degenerate subgraph of $G \cup H$. Thus $\varrho_k(G \cup H) \leqslant (k+1)ts$, and the lemma is proved.

THEOREM 5. $M_k(G) + M_k(H) \leqslant |V| + (k+1) M_k(G \cup H)$.

Proof. Let $A, B \subset V$ be the sets which induce k-degenerate subgraphs in G and H, respectively, and let $|A| = M_k(G)$ and $|B| = M_k(H)$. Now we get

$$|A| + |B| = |A \cup B| + |A \cap B| \leqslant |V| + |A \cap B|.$$

From the left inequality of Corollary 1 we obtain

$$\text{(a)} \qquad M_k(G) + M_k(H) \leqslant |V| + \varrho_k(\langle A \cap B \rangle) M_k(\langle A \cap B \rangle).$$

Since $\langle A \cap B \rangle$ is a subgraph of $G \cup H$, we have

$$\text{(b)} \qquad M_k(\langle A \cap B \rangle) \leqslant M_k(G \cup H).$$

Obviously, $\varrho_k(\langle A \rangle) = \varrho_k(\langle B \rangle) = 1$. Thus, by Lemma 2, we obtain

$$\text{(c)} \qquad \varrho_k(\langle A \cap B \rangle) \leqslant \varrho_k(\langle A \rangle \cup \langle B \rangle) \leqslant k+1.$$

The theorem now follows from (a), (b), and (c).

COROLLARY 3. $M_k(G) + M_k(\bar{G}) \leqslant |V| + (k+1)^2$.

Lick and White (see [1]) generalized the Nordhaus–Gaddum theorem by proving the following

PROPOSITION 2. *For a nonnegative integer k and a graph G of order n,*

$$\frac{\sqrt{n}}{k+1} \leqslant \sqrt{\varrho_k(G)\varrho_k(\bar{G})} \leqslant \frac{1}{2}(\varrho_k(G) + \varrho_k(\bar{G})) \leqslant \frac{n+2k+1}{2k+2}.$$

In [14] Schürger proved the following

PROPOSITION 3. *For any graphs G and H we have*

(i) $\varrho_0(G) + \varrho_0(H) \leqslant |V| + \varrho_0(G \cap H)$,

(ii) $\varrho_0(G \cup H) \leqslant \varrho_0(G)\varrho_0(H)$.

A generalization of Propositions 2 and 3 will be proved below.

THEOREM 6. *For a nonnegative integer k and graphs G, H of order n,*

$$\frac{\sqrt{\varrho_0(G\cup H)}}{k+1} \leqslant \sqrt{\varrho_k(G)\varrho_k(H)} \leqslant \frac{1}{2}\big(\varrho_k(G)+\varrho_k(H)\big) \leqslant \frac{n+\varrho_0(G\cap H)(2k+1)}{2k+2}.$$

Proof. From Corollary 2 for $r = k$ we obtain

$$\frac{1}{k+1}\varrho_0(G) \leqslant \varrho_k(G) \quad\text{and}\quad \frac{1}{k+1}\varrho_0(H) \leqslant \varrho_k(H).$$

Hence

$$\text{(a)}\qquad \frac{1}{(k+1)^2}\varrho_0(G)\varrho_0(H) \leqslant \varrho_k(G)\varrho_k(H).$$

Thus, by Proposition 3 (ii) and (a), we obtain the left inequality in the statement of the theorem.

To show the right inequality, it is obviously sufficient to consider the case where $G\cup H$ is a complete graph.

Let $V = V_1 \cup \ldots \cup V_t$ be a partition of V into nonempty sets V_i being independent in $\langle E(G\cap H)\rangle$. Obviously,

$$\varrho_0(\langle E(G\cap H)\rangle) = t = \varrho_0(G\cap H).$$

Let E_i and F_i denote the sets of all lines of $E(G)\backslash E(H)$ and $E(H)\backslash E(G)$, respectively, whose end-points belong to V_i, $1 \leqslant i \leqslant t$. The graphs $G_i = (V_i, E_i)$ and $H_i = (V_i, F_i)$ satisfy the condition $\bar{G}_i = H_i$ for $i = 1, \ldots, t$. Thus Proposition 2 gives

$$\text{(b)}\qquad \varrho_k(G_i)+\varrho_k(H_i) \leqslant \frac{n_i+2k+1}{k+1},$$

where $n_i = |V_i|$, $1 \leqslant i \leqslant t$. Obviously,

$$\sum_{i=1}^{t} \varrho_k(G_i) \geqslant \varrho_k(G) \quad\text{and}\quad \sum_{i=1}^{t} \varrho_k(H_i) \geqslant \varrho_k(H).$$

Consequently, by (b), we obtain

$$\varrho_k(G)+\varrho_k(H) \leqslant \sum_{i=1}^{t} \frac{n_i+2k+1}{k+1} = \frac{n+t(2k+1)}{k+1}$$

and the theorem is proved.

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