COLLOQUIUM MATHEMATICUM

VOL. XXXIII 1975 FASC. 1

QUASI-COMPLETE IDEAL LATTICES

BY

JOHNNY A. JOHNSON (HOUSTON, TEXAS)

1. Introduction. In [3] a topology was introduced on the set $S(R)$ of ideals of a local ring R and this topology was used to study a class of local rings which satisfy a well-known property of complete local rings. This property was called *quasi-completeness*. In this note we are interested in the lattice-theoretical properties of the lattice of ideals of a quasi-complete local ring and we characterize such rings by properties of these lattices. One characterization obtained is that R is quasi-complete if and only if the Noether lattice of ideals of R and the Noether lattice of ideals of its completion are isomorphic as multiplicative lattices. Thus, for purely ideal-theoretic purposes, one can assume that a quasi-complete local ring is (topologically) complete.

2. Quasi-completeness. We shall in general adopt the ring terminology of [5]. In particular, all rings are assumed to be commutative with a multiplicative identity. We denote the M-adic ring completion of the local ring (R, M) by (R^*, M^*), or just by R^*, and, for an ideal A of R, we will use AR^* to denote the ideal generated by A in R^*. The Noether lattice terminology will be that of [1].

A local ring (R, M) is said to be *quasi-complete* if, whenever $\langle B_i\rangle$, $i = 1, 2, \ldots$, is a decreasing sequence of ideals of R and n is a positive integer, then there exists an integer $s(n)$ such that

$$B_{s(n)} \subseteq \Big(\bigwedge_{i=1}^{\infty} B_i\Big) + M^n.$$

Remark 1. If a local ring (R, M) is complete in its M-topology, then it is quasi-complete (cf. [6], Theorem 1, p. 86). The converse is not true (see the Example).

If we let $\mathfrak{L}(R)$ denote the multiplicative lattice of ideals of a local ring (R, M), then it is known that $\mathfrak{L}(R)$ is a local Noether lattice (cf. [1], p. 486) with maximal element M. A metric, called the *M-adic metric*, can be defined on $\mathfrak{L}(R)$ (cf. [3] and Section 3 of [4]) as follows:

for each A, B in $\mathfrak{L}(R)$, set

$$S(A, B) = \operatorname{Sup}\{i\colon A+M^i = B+M^i\} \qquad (\text{where } M^0 = R)$$

and

$$d_M(A, B) = 2^{-S(A,B)}.$$

THEOREM. *Let (R, M) be a local ring. Then the following four statements are equivalent*:

(i) *$\mathfrak{L}(R)$ is complete with the M-adic metric.*

(ii) *R is quasi-complete.*

(iii) *If $\langle A_i\rangle$, $i = 1, 2, \ldots$, is a decreasing sequence in $\mathfrak{L}(R)$, then*

$$A_i \to \bigcap_{i=1}^{\infty} A_i \quad \textit{as } i \to \infty$$

in the M-adic metric.

(iv) *$\mathfrak{L}(R)$ and $\mathfrak{L}(R^*)$ are isomorphic as multiplicative lattices.*

Proof. (i) and (ii) are equivalent by Theorem 1 of [3], p. 197. If $\langle B_i\rangle$ is a decreasing sequence in $\mathfrak{L}(R)$ and B is an element of $\mathfrak{L}(R)$ with $B \leqslant \bigcap_{i=1}^{\infty} A_i$, then $\langle A_i\rangle$ d_M-converges to B if and only if, for each integer $n \geqslant 0$, there is an integer $s(n) \geqslant 1$ such that

$$A_i+M^n = B+M^n \quad \text{for all integers } i \geqslant s(n),$$

and so (ii) and (iii) are equivalent.

If R is quasi-complete, then $\varphi\colon A \to AR^*$ maps $\mathfrak{L}(R)$ onto $\mathfrak{L}(R^*)$. Furthermore φ is injective (cf. [5], (17.9), p. 57) and a multiplicative lattice homomorphism so that (ii) implies (iv). Assume now that there is a multiplicative lattice isomorphism $\varphi\colon \mathfrak{L}(R^*) \to \mathfrak{L}(R)$. Then, by the lattice-preserving properties of φ, we have

$$\varphi(M^{*n}) = M^n \quad \text{for all integers } n \geqslant 0.$$

Thus, for each integer $n \geqslant 0$ and for all elements A and B of $\mathfrak{L}(R^*)$, we obtain

$$A \vee M^{*n} = B \vee M^{*n} \quad \text{if and only if} \quad \varphi(A) \vee M^n = \varphi(B) \vee M^n.$$

Hence, for all A and B in $\mathfrak{L}(R^*)$, we have

(1) $$d_{M^*}(A, B) = d_M\big(\varphi(A), \varphi(B)\big).$$

Since R^* is quasi-complete (Remark 1), it follows from the equivalence of (i) and (ii) that $\mathfrak{L}(R^*)$ is d_{M^*}-complete, and this combined with (1) shows that $\mathfrak{L}(R)$ is d_M-complete. Thus (iv) implies (i) and the proof is complete.

The analytical irreducibility of a local ring (R, M) and the M-adic metric completeness of the Noether lattice $\mathfrak{L}(R)$ are related as follows:

COROLLARY 1. *Let (R, M) be a local ring such that $\mathfrak{L}(R)$ is complete with the M-adic metric. In order that R be analytically irreducible it is necessary and sufficient that the zero element of $\mathfrak{L}(R)$ be prime.*

The proof follows from (iv) of the Theorem.

COROLLARY 2. *Every quasi-complete local domain is analytically irreducible.*

Example. Let Z denote the rational integers, let p be a prime element of Z, and consider the local ring $(Z_{(p)}, pZ_{(p)})$. Clearly, $Z_{(p)}$ is not complete in its $pZ_{(p)}$-adic topology, but it is easily verified that it is, in fact, quasi--complete. The details are omitted. We note in passing that this observation (together with the Theorem) and known properties of the p-adic integers show that the local Noether lattice $\mathfrak{L}(Z_{(p)})$ and the Noether lattice $\mathfrak{L}(Z^*)$ of ideals of the p-adic ring completion of Z are isomorphic as multiplicative lattices.

Before proceeding further, we pause to collect under one heading several simple properties of local rings which may not be Noetherian (see [5], p. 13, for the definition) and which have principal maximal elements. The proof is straightforward and is omitted.

LEMMA. *Let (R, M) be a local ring which may not be Noetherian. If M is principal, then (R, M) satisfies the following:*

(i) *Every proper ideal of R is a power of M.*

(ii) *Every ideal of R is principal.*

(iii) *The ideals of R are linearly ordered.*

(iv) *(R, M) is a local ring (i.e., R is Noetherian).*

PROPOSITION. *Let (R, M) be a local ring which may not be Noetherian. If M is principal, then (R, M) is a quasi-complete local ring.*

Proof. This follows in a straightforward manner from the Lemma and (iii) of the Theorem. We omit the details.

COROLLARY 3. *Let (R, M) be a local ring. If the maximal element of $\mathfrak{L}(R)$ is a meet-principal element of $\mathfrak{L}(R)$, then (R, M) is quasi-complete.*

Proof. In the local ring case it is known ([2], Corollary 1.6, p. 127) that the principal ideals of R and the principal elements of $\mathfrak{L}(R)$ coincide. It is also known (and easily verified) that meet-principal and principal are equivalent concepts in any Noether lattice. These two facts in conjunction with the Proposition complete the proof.

Remark 2. Since the maximal ideal of a Noetherian valuation ring is principal, it follows from the Proposition that such a ring is quasi-complete and, therefore, by the Theorem, that the completion R^* of a Noetherian valuation ring R is a Noetherian valuation ring of the same dimension. In particular, the completion of a rank one discrete valuation ring is a rank one discrete valuation ring.

The analytical unramifiedness of a local ring (R, M) and the M-adic metric completeness of the Noether lattice $\mathfrak{L}(R)$ are related as follows:

Remark 3. *Let (R, M) be a local ring such that $\mathfrak{L}(R)$ is complete with the M-adic metric. A necessary and sufficient condition that R be analytically unramified is that $\mathfrak{L}(R)$ have no non-trivial meet-principal nilpotent elements.*

Proof. Keeping in mind that $\mathfrak{L}(R)$ and $\mathfrak{L}(R^*)$ are isomorphic as multiplicative lattices (see the Theorem) one proceeds as in the proof of Corollary 3. We omit the details.

COROLLARY 4. *Every quasi-complete local ring without proper nilpotent elements is analytically unramified.*

REFERENCES

[1] R. P. Dilworth, *Abstract commutative ideal theory*, Pacific Journal of Mathematics 12 (1962), p. 481-498.

[2] E. W. Johnson, *A-transforms and Hilbert functions on local lattices*, Transactions of the American Mathematical Society 137 (1969), p. 125-139.

[3] — *A note on quasi-complete local rings*, Colloquium Mathematicum 21 (1970), p. 197-198.

[4] J. A. Johnson, *A-adic completions of Noetherian lattice modules*, Fundamenta Mathematicae 66 (1970), p. 347-373.

[5] M. Nagata, *Local rings*, New York 1962.

[6] D. G. Northcott, *Ideal theory*, Cambridge 1963.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HOUSTON
HOUSTON, TEXAS

Reçu par la Rédaction le 3. 9. 1973