COLLOQUIUM MATHEMATICUM

VOL. LV 1988 FASC. 1

QUASI-REGULAR GENERALIZED CONVOLUTIONS

BY

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1. Notation and preliminaries. Generalized convolutions were introduced in [3]. Let us recall some definitions. We denote by $\mathfrak{P}$ the set of all probability measures defined on Borel subsets of the positive half-line R_+. The set $\mathfrak{P}$ is endowed with the topology of weak convergence. For $\mu \in \mathfrak{P}$ and $a > 0$ we define the map T_a by setting $(T_a \mu)(E) = \mu(a^{-1} E)$ for all Borel subsets E of R_+. By δ_c we denote the probability measure concentrated at the point c.

A continuous in each variable separately commutative and associative $\mathfrak{P}$-valued binary operation $\circ$ on $\mathfrak{P}$ is called a *generalized convolution* if it is distributive with respect to convex combinations and maps T_a $(a > 0)$ with δ_0 as the unit element. Moreover, the key axiom postulates the existence of norming constants c_n and a measure $\gamma \in \mathfrak{P}$ different from δ_0 such that

$$T_{c_n} \delta_1^{\circ n} \to \gamma, \tag{1.1}$$

where $\delta_1^{\circ n}$ is the n-th power of δ_1 under $\circ$. The set $\mathfrak{P}$ with the operation $\circ$ and all operations of convex combinations is called a *generalized convolution algebra* and denoted by $(\mathfrak{P}, \circ)$. A generalized convolution algebra is said to be *quasi-regular* if the norming sequence c_n in condition (1.1) tends to 0. This concept was introduced by Kucharczak in [1].

For any pair $\mu, \nu \in \mathfrak{P}$ we denote by $\mu \Box \nu$ the probability distribution of $\max(X, Y)$ where the random variables X and Y are independent and have the probability distributions μ and ν, respectively. It is clear that $(\mathfrak{P}, \Box)$ is a generalized convolution algebra. Moreover,

$$\delta_a \Box \delta_b = \delta_{\max(a,b)} \quad (a, b \in R_+). \tag{1.2}$$

Using formula (1.2) one can easily show that $(\mathfrak{P}, \Box)$ is not quasi-regular. The aim of this paper is to prove that $(\mathfrak{P}, \Box)$ is the only non-quasi-regular generalized convolution algebra. The main results of this paper are based on two techniques: one uses semigroup method, the other uses compactness arguments for probability measures on the compactified half-line.

2. Extended generalized convolution algebras. An extension of a generalized convolution to the space $\bar{\mathfrak{P}}$ of all probability measures on compactified half-line $\bar{R}_+ = [0, \infty]$ enables us to use compactness arguments, and therefore is a useful tool in the study of generalized convolutions. The space $\bar{\mathfrak{P}}$ is compact in the topology of weak convergence of probability measures. We identify the space $\mathfrak{P}$ with the subspace of $\bar{\mathfrak{P}}$ consisting of measures with zero mass at ∞. Further, we introduce the notation $\mathfrak{P}_\infty = \bar{\mathfrak{P}} \setminus \mathfrak{P}$.

Each measure $\mu \in \bar{\mathfrak{P}}$ has the unique representation

$$\mu = a\mu' + (1-a)\delta_\infty, \quad \text{where } \mu' \in \mathfrak{P} \text{ and } 0 \leqslant a \leqslant 1.$$

Given a generalized convolution algebra $(\mathfrak{P}, \circ)$ we extend the operations $\circ$ and T_c $(c > 0)$ on $\bar{\mathfrak{P}}$ by setting

$$(2.1) \qquad \big(a\mu' + (1-a)\delta_\infty\big) \circ \big(b\nu' + (1-b)\delta_\infty\big) = ab(\mu' \circ \nu') + (1-ab)\delta_\infty$$

and

$$(2.2) \qquad T_c\big(a\mu' + (1-a)\delta_\infty\big) = aT_c\mu' + (1-a)\delta_\infty,$$

where $0 \leqslant a, b \leqslant 1$, $\mu', \nu' \in \mathfrak{P}$ and $c > 0$. The set $\bar{\mathfrak{P}}$ with the operations $\circ$ and convex combinations is called the *extended generalized convolution algebra* and denoted by $(\bar{\mathfrak{P}}, \circ)$. It is clear that the algebraic properties of $(\mathfrak{P}, \circ)$ carry over to $(\bar{\mathfrak{P}}, \circ)$. Namely, $\circ$ is a commutative semigroup operation on $\bar{\mathfrak{P}}$ distributive with respect to convex combinations and maps T_c $(c > 0)$. The measure δ_0 is the unit element in $\bar{\mathfrak{P}}$. Moreover, by (2.2), we have the following simple statements:

PROPOSITION 2.1. *If $\mu_n, \mu \in \bar{\mathfrak{P}}$, $\mu_n \to \mu$, $\mu(\{0\}) = 0$ and $c_n \to \infty$, then $T_{c_n}\mu_n \to \delta_\infty$.*

PROPOSITION 2.2. *If $\mu_n, \mu \in \bar{\mathfrak{P}}$, $\mu_n \to \mu$, $\mu \neq \delta_0$ and $c_n \to \infty$, then all limit points of the sequence $T_{c_n}\mu_n$ belong to $\mathfrak{P}_\infty$.*

The continuity properties of $\circ$ on $\bar{\mathfrak{P}}$ will be discussed later.

Let Y be a locally compact metric space and m a Borel probability measure on Y. For any $\bar{\mathfrak{P}}$-valued continuous function $\nu(\cdot)$ on Y the integral

$$\mu = \int_Y \nu(y)\, m(dy)$$

is taken in the weak sense, i.e.,

$$\int_{\bar{R}_+} f(x)\mu(dx) = \int_Y \int_{\bar{R}_+} f(x)\nu(y)(dx)\, m(dy)$$

for every continuous function f on $\bar{R}_+$. By $N(\lambda)$ we denote the support of the measure λ. By the continuity of the function $\nu(\cdot)$ we have the following useful remark:

PROPOSITION 2.3. *If $\mu = \int_Y \nu(y)\, m(dy)$, then*

$$N(\nu(y)) \subset N(\mu) \quad \textit{for all } y \in N(m).$$

The Laplace transform of $\mu \in \mathfrak{P}$ is defined by the formula

$$\tilde{\mu}(z) = \int_{R_+} e^{-zx} \mu(dx) \quad (0 < z < \infty).$$

The following simple properties will be useful:

$$\tilde{\mu}(\infty) = \mu(\{0\}), \tag{2.3}$$

$$\tilde{\mu}(0+) = \mu(R_+), \tag{2.4}$$

$$(T_c \mu)^{\sim}(z) = \tilde{\mu}(cz) \quad (c, z \in (0, \infty)), \tag{2.5}$$

$$\mu_n \to \mu \text{ yields } \tilde{\mu}_n \to \tilde{\mu} \tag{2.6}$$

uniformly on every compact subset of $(0, \infty)$. Conversely, if $\tilde{\mu}_n \to \tilde{\mu}$ pointwise on $(0, \infty)$, then $\mu_n \to \mu$.

LEMMA 2.1. *Let $\nu_n(\cdot)$ be a sequence of $\mathfrak{P}$-valued continuous functions on a locally compact metric space Y with a Borel probability measure m. If*

$$\int_Y \nu_n(y)\, m(dy) \to \delta_0,$$

then there exists a subsequence $n_1 < n_2 < \ldots$ such that $\nu_{n_k}(y) \to \delta_0$ for m-almost all y.

Proof. Put $\mu_n = \int_Y \nu_n(y) m(dy)$. Then

$$\int_Y \tilde{\nu}_n(y)(z)\, m(dy) = \tilde{\mu}_n(z) \to 1.$$

Since $0 \leqslant \tilde{\nu}_n(y)(z) \leqslant 1$, the last relation yields the convergence $\tilde{\nu}_n(y)(z) \to 1$ in probability m. Thus for a fixed $z_0 \in (0, \infty)$ we can find a subsequence $n_1 < n_2 < \ldots$ such that $\tilde{\nu}_{n_k}(y)(z_0) \to 1$ for m-almost all y. But this is possible only if $\nu_{n_k}(y) \to \delta_0$ for m-almost all y, which completes the proof.

First we shall establish continuity properties of $\delta_t \circ \delta_u$ $(t, u \in \bar{R}_+)$. Put

$$h(t, u, z) = (\delta_t \circ \delta_u)^{\sim}(z) \quad (z \in (0, \infty),\ t, u \in \bar{R}_+).$$

The function h has the following properties:

$$0 \leqslant h(t, u, z) \leqslant 1, \tag{2.7}$$

$$h(t, u, z) = h(u, t, z), \tag{2.8}$$

$$h(at, au, z) = h(u, t, az) \quad (a > 0), \tag{2.9}$$

$$h(t, 0, z) = e^{-zt}, \tag{2.10}$$

$$h(\infty, u, z) = 0. \tag{2.11}$$

Moreover, by (2.6),

$$h(t_n, u, z_n) \to h(t, u, z) \tag{2.12}$$

when $t_n \to t$, $z_n \to z$ $(t, u, z \in (0, \infty))$.

LEMMA 2.2. *For every $z \in (0, \infty)$ the function $h(\cdot, \cdot, z)$ is continuous at every point different from (∞, ∞).*

Proof. Let $t_n \to t$, $u_n \to u$ and $(t, u) \neq (\infty, \infty)$. By (2.8) without loss of generality we may assume that $t \geqslant u$ and $t_n \geqslant u_n$ $(n = 1, 2, \ldots)$.

First consider the case $t = \infty$. Then $0 \leqslant u < \infty$. Moreover, we may assume that $t_n > 0$. Setting $v_n = 0$ if $t_n = \infty$ and $v_n = u_n/t_n$ if $t_n < \infty$, we have $v_n \to 0$ and, by (2.9) and (2.11),

$$h(t_n, u_n, z) = 0 \quad \text{if } t_n = \infty$$

and

$$h(t_n, u_n, z) = h(1, v_n, t_n z) \quad \text{if } t_n < \infty.$$

Since the Laplace transform is monotone non-increasing and, for every integer k, $t_n z > k$ for n large enough, we have the inequality $h(t_n, u_n, z) \leqslant h(1, v_n, k)$ for n large enough. Thus, by (2.10) and (2.12), we have the inequality

$$\overline{\lim_{n\to\infty}}\, h(t_n, u_n, z) \leqslant h(1, 0, k) = e^{-k} \quad (k = 1, 2, \ldots),$$

which, by (2.7) and (2.11), yields

$$\lim_{n\to\infty} h(t_n, u_n, z) = 0 = h(\infty, u, z).$$

This shows that $h(\cdot, \cdot, z)$ is continuous at the points (∞, u) $(0 \leqslant u < \infty)$.

Suppose now that $t = 0$. Then we have also $u = 0$. Put $v_n = 0$ if $t_n = 0$ and $v_n = u_n/t_n$ if $t_n > 0$. Given $\varepsilon > 0$, we have $t_n z < \varepsilon$ for n large enough. Consequently, by (2.7), (2.9) and (2.10), for n large enough we obtain

$$h(t_n, u_n, z) = 1 \geqslant h(1, v_n, \varepsilon) \quad \text{if } t_n = 0$$

and

$$h(t_n, u_n, z) = h(1, v_n, t_n z) \geqslant h(1, v_n, \varepsilon) \quad \text{if } t_n > 0.$$

For a fixed z, taking a subsequence $n_1 < n_2 < \ldots$ such that

$$\lim_{k\to\infty} h(t_{n_k}, u_{n_k}, z) = \underline{\lim}_{n\to\infty}\, h(t_n, u_n, z)$$

we may assume in addition that $v_{n_k} \to v$ $(0 \leqslant v \leqslant 1)$. Thus

$$\lim_{k\to\infty} h(t_{n_k}, u_{n_k}, z) \geqslant h(1, v, \varepsilon)$$

for every $\varepsilon > 0$, which yields

$$\lim_{n\to\infty} h(t_n, u_n, z) \geqslant h(1, v, 0+).$$

Taking into account (2.4) we conclude that

$$h(1, v, 0+) = (\delta_1 \circ \delta_v)(R_+) = 1,$$

which, by (2.7), implies

$$\lim_{n\to\infty} h(t_n, u_n, z) = 1 = h(0, 0, z).$$

Consequently, the function $h(\cdot, \cdot, z)$ is continuous at the point (0, 0).

It remains the case $0 < t < \infty$. Of course, we may assume that $0 < t_n < \infty$. Setting $v_n = u_n/t_n$ we have, by (2.9) and (2.12),

$$h(t_n, u_n, z) = h(1, v_n, t_n, z) \to h(1, u/t, z) = h(t, u, z),$$

which completes the proof.

By relation (2.6) and Lemma 2.2 we infer that the $\mathfrak{P}$-valued function $\delta_t \circ \delta_u$ $((t, u) \in R_+ \times R_+)$ is continuous and, consequently, the integral

$$\int_{R_+} \int_{R_+} \delta_t \circ \delta_u \, \mu(dt) \, \nu(du)$$

exists for any pair $\mu, \nu \in \mathfrak{P}$. It is easy to check the formula

$$\mu \circ \nu = \int_{R_+} \int_{R_+} \delta_t \circ \delta_u \, \mu(dt) \, \nu(du) \qquad (\mu, \nu \in \mathfrak{P}). \tag{2.13}$$

Indeed, this formula is obvious if μ and ν are concentrated at a finite number of points. Then we use the continuity of $\circ$ in each variable separately. As a consequence of definition (2.1) and formulas (2.11) and (2.13) we get the following

COROLLARY 2.1. *We have*

$$(\mu \circ \nu)^{\sim}(z) = \int_{R_+} \int_{R_+} h(t, u, z) \, \mu(dt) \, \nu(du) \qquad (\mu, \nu \in \bar{\mathfrak{P}}).$$

The continuity properties of $\circ$ on $\bar{\mathfrak{P}}$ are described by the following statement:

PROPOSITION 2.4. *The generalized convolution $\circ$ is continuous in both variables at all points $(\mu, \nu) \notin \mathfrak{P}_\infty \times \mathfrak{P}_\infty$.*

Proof. If $(\mu, \nu) \notin \mathfrak{P}_\infty \times \mathfrak{P}_\infty$, then the product measure $\mu \times \nu$ on $\bar{R}_+ \times \bar{R}_+$ has zero mass at the point (∞, ∞). Suppose that $\mu_n \to \mu$ and $\nu_n \to \nu$. Then, of course, $\mu_n \times \nu_n \to \mu \times \nu$ and, by Lemma 2.2 and inequality (2.7), we have the relation

$$\int_{\bar{R}_+} \int_{\bar{R}_+} h(t, u, z) \, \mu_n(dt) \, \nu_n(du) \to \int_{\bar{R}_+} \int_{\bar{R}_+} h(t, u, z) \, \mu(dt) \, \nu(du).$$

In other words, by Corollary 2.1, $(\mu_n \circ \nu_n)^{\sim} \to (\mu \circ \nu)^{\sim}$ pointwise, which, by (2.6), yields the continuity of $\circ$ at the point (μ, ν). The proposition is thus proved.

From Proposition 2.4 it follows immediately

THEOREM 2.1. *The operation* $\circ$ *in a generalized convolution algebra* $(\mathfrak{P}, \circ)$ *is continuous in both variables.*

Now we shall discuss the continuity properties of $\circ$ on $\mathfrak{P}_\infty \times \mathfrak{P}_\infty$. The following example shows that $\circ$ may be not continuous in both variables on $\mathfrak{P}_\infty \times \mathfrak{P}_\infty$.

EXAMPLE. Consider the generalized convolution defined by the relation

$$\delta_a \circ \delta_b = \tfrac{1}{2}(\delta_{|a-b|} + \delta_{a+b}) \quad (a, b \in R_+)$$

([3], p. 218). Put $\mu_{2k} = \delta_k$ and $\mu_{2k+1} = \delta_{2k+1}$ $(k = 1, 2, \ldots)$. Then $\mu_n \to \delta_\infty$ and $\delta_n \to \delta_\infty$. But

$$\mu_{2k} \circ \delta_{2k} = \tfrac{1}{2}(\delta_k + \delta_{3k}), \quad \mu_{2k+1} \circ \delta_{2k+1} = \tfrac{1}{2}(\delta_0 + \delta_{4k+2}) \quad (k = 1, 2, \ldots),$$

which shows that the sequence $\mu_n \circ \delta_n$ is not convergent and has two limit points δ_∞ and $\frac{1}{2}(\delta_0 + \delta_\infty)$.

First we shall establish some lemmas.

LEMMA 2.3. *If* $\mu, \nu \in \bar{\mathfrak{P}}$ *and* $\mu \circ \nu = \delta_0$, *then* $\mu = \nu = \delta_0$.

Proof. By definition (2.1) we infer that $\mu, \nu \in \mathfrak{P}$. Applying formula (2.13) we have

$$\int_{R_+} \int_{R_+} \delta_t \circ \delta_u \, \mu(dt) \, \nu(du) = \delta_0,$$

which, by Proposition 2.3, yields

$$N(\delta_t \circ \delta_u) \subset \{0\} \quad \text{for } (t, u) \in N(\mu) \times N(\nu).$$

In other words, $\delta_t \circ \delta_u = \delta_0$ for $(t, u) \in N(\mu) \times N(\nu)$. If either μ or ν is not concentrated at 0, then $\delta_a \circ \delta_b = \delta_0$ for a certain pair $a \leqslant b$, $b > 0$. Furthermore, applying the map $T_{b^{-1}}$ to the last equation and setting $c = a/b$, we get $\delta_1 \circ \delta_c = \delta_0$. Then $\delta_c \circ \delta_{c^2} = \delta_0$, which yields $\delta_1 = \delta_1 \circ \delta_c \circ \delta_{c^2} = \delta_{c^2}$. Consequently, $c = 1$ and $\delta_1 \circ \delta_1 = \delta_0$. Taking the norming sequence c_n in (1.1) we have

$$T_{c_{2n}} \delta_1^{\circ 2n} \to \gamma \neq \delta_0.$$

On the other hand, $\delta_1^{\circ 2n} = \delta_0$, which yields a contradiction. The lemma is thus proved.

COROLLARY 2.2. *We have*

$$\sup \{(\delta_c \circ \delta_1)(\{0\}): \; 0 \leqslant c \leqslant 1\} < 1.$$

To prove this it is enough to show that

$$\overline{\lim_{n \to \infty}} \, (\delta_{c_n} \circ \delta_1)(\{0\}) < 1$$

for any convergent sequence c_n $(0 \leqslant c_n \leqslant 1)$. Suppose the contrary, i.e.,

$$\overline{\lim_{n \to \infty}} (\delta_{c_n} \circ \delta_1)(\{0\}) = 1$$

for a certain sequence c_n tending to c $(0 \leqslant c \leqslant 1)$. Then $(\delta_c \circ \delta_1)(\{0\}) = 1$ or, equivalently, $\delta_c \circ \delta_1 = \delta_0$, which contradicts Lemma 2.3.

LEMMA 2.4. *If* $g(z) = \overline{\lim_{(t,u) \to (\infty, \infty)}} h(t, u, z)$, *then*

$$\sup\{g(z) \colon z \in (0, \infty)\} < 1.$$

Proof. Given $z \in (0, \infty)$, we can choose a sequence $(t_n, u_n) \to (\infty, \infty)$ such that

$$g(z) = \lim_{n \to \infty} h(t_n, u_n, z).$$

Moreover, by (2.8), we may assume that $0 < u_n \leqslant t_n < \infty$ $(n = 1, 2, \ldots)$. Setting $c_n = u_n/t_n$ and passing to a subsequence if necessary, we may assume without loss of generality that $c_n \to c$ $(0 \leqslant c \leqslant 1)$. Further, by (2.9), we have

$$h(t_n, u_n, z) = h(1, c_n, t_n z).$$

For any integer k we have $t_n z > k$ if n is large enough. Consequently, the last equation implies the inequality

$$h(t_n, u_n, z) \leqslant h(1, c_n, k)$$

for sufficiently large n. Thus

$$g(z) \leqslant h(1, c, k) \quad (k = 1, 2, \ldots),$$

which, by (2.3), yields $g(z) \leqslant (\delta_1 \circ \delta_c)(\{0\})$. Now our assertion is a direct consequence of Corollary 2.2.

LEMMA 2.5. *If* $\mu_n, \nu_n \in \bar{\mathfrak{P}}$ *and* $\mu_n \to \delta_\infty$, $\nu_n \to \delta_\infty$, *then all limit points of* $\mu_n \circ \nu_n$ *belong to* $\mathfrak{P}_\infty$.

Proof. The relations $\mu_n \to \delta_\infty$ and $\nu_n \to \delta_\infty$ imply the existence of a sequence $(t_n, u_n) \to (\infty, \infty)$ such that

$$\mu_n([0, t_n]) \to 0 \quad \text{and} \quad \nu_n([0, u_n]) \to 0.$$

Using Corollary 2.1 we have the inequality

$$(\mu_n \circ \nu_n)^{\sim}(z) \leqslant \mu_n([0, t_n]) + \nu_n([0, u_n]) + \sup\{h(t, u, z) \colon t \geqslant t_n, u \geqslant u_n\}.$$

Thus

$$\overline{\lim_{n \to \infty}} (\mu_n \circ \nu_n)^{\sim}(z) \leqslant g(z) \quad (z \in (0, \infty))$$

and, consequently, for any limit point λ of the sequence $\mu_n \circ \nu_n$ the inequality $\tilde{\lambda}(z) \leqslant g(z)$ $(z \in (0, \infty))$ is true. Applying Lemma 2.4 we have the inequality

$\lambda(R_+) = \tilde{\lambda}(0+) < 1$, which shows that λ has a positive mass at ∞. Thus $\lambda \in \mathfrak{P}_\infty$, which completes the proof.

PROPOSITION 2.5. *Suppose that* $\mu_n, \nu_n \in \bar{\mathfrak{P}}$, $\mu, \nu \in \mathfrak{P}_\infty$ *and* $\mu_n \to \mu$, $\nu_n \to \nu$. *Then all limit points of* $\mu_n \circ \nu_n$ *belong to* $\mathfrak{P}_\infty$.

Proof. Suppose that

$$\mu = a\mu' + (1-a)\delta_\infty, \qquad \nu = b\nu' + (1-b)\delta_\infty,$$

where $\mu', \nu' \in \mathfrak{P}$ and

$$0 \leqslant a < 1, \quad 0 \leqslant b < 1. \tag{2.14}$$

Then the sequences μ_n, ν_n have a representation

$$\mu_n = a_n \mu_n' + (1-a_n)\mu_n'', \qquad \nu_n = b_n \nu_n' + (1-b_n)\nu_n'',$$

where $a_n \to a$, $b_n \to b$, $\mu_n', \nu_n' \in \mathfrak{P}$, $\mu_n' \to \mu'$, $\nu_n' \to \nu'$, $\mu_n'' \to \delta_\infty$, and $\nu_n'' \to \delta_\infty$. Since

$$\mu_n \circ \nu_n = a_n b_n \mu_n' \circ \nu_n' + (1-a_n) b_n \mu_n'' \circ \nu_n' + a_n(1-b_n)\mu_n' \circ \nu_n'' + (1-a_n)(1-b_n)\mu_n'' \circ \nu_n''$$

and, by Lemma 2.5, all limit points of $\mu_n'' \circ \nu_n''$ belong to $\mathfrak{P}_\infty$, we conclude, by virtue of (2.14), that all limit points of $\mu_n \circ \nu_n$ have positive mass at ∞. The proposition is thus proved.

COROLLARY 2.3. *If* $\mu_n, \nu_n \in \bar{\mathfrak{P}}$, $\lambda \in \mathfrak{P}$ *and* $\mu_n \circ \nu_n \to \lambda$, *then all limit points of the sequences* μ_n *and* ν_n, *respectively, belong to* $\mathfrak{P}$.

Proof. It is enough to show that each limit point of μ_n belongs to $\mathfrak{P}$. Let $\mu_{n_k} \to \mu$. Passing to a subsequence if necessary we may assume without loss of generality that the sequence ν_{n_k} is also convergent, say to ν. By Proposition 2.5, $(\mu, \nu) \notin \mathfrak{P}_\infty \times \mathfrak{P}_\infty$. Consequently, by Proposition 2.4, the operation $\circ$ is continuous at the point (μ, ν). Thus $\mu \circ \nu = \lambda$, which, by definition (2.1), shows that $\mu \in \mathfrak{P}$. This completes the proof.

COROLLARY 2.4. *If* $\mu_n, \nu_n \in \bar{\mathfrak{P}}$ *and* $\mu_n \circ \nu_n \to \delta_0$, *then* $\mu_n \to \delta_0$ *and* $\nu_n \to \delta_0$.

Proof. Let $\mu_{n_k} \to \mu$. Passing to a subsequence if necessary, we may assume that $\nu_{n_k} \to \nu$. By Corollary 2.3, $\mu, \nu \in \mathfrak{P}$ and, consequently, by Proposition 2.4, the operation $\circ$ is continuous at the point (μ, ν). Thus $\mu \circ \nu = \delta_0$, which, by Lemma 2.3, yields $\mu = \nu = \delta_0$. This proves the relation $\mu_n \to \delta_0$. By symmetry of our assumptions we have also $\nu_n \to \delta_0$, which completes the proof.

COROLLARY 2.5. *If*

$$\mathop{\bigcirc}_{j=1}^{n_k} \delta_{u_{j,k}} \to \delta_0,$$

then

$$v_k = \max\{u_{j,k}\colon j = 1, 2, \ldots, n_k\} \to 0.$$

Proof. Since

$$\mathop{\circ}_{j=1}^{n_k} \delta_{u_{j,k}} = \delta_{v_k} \circ \varrho_k,$$

where $\varrho_k \in \bar{\mathfrak{P}}$, we have, by virtue of Corollary 2.4, $\delta_{v_k} \to \delta_0$, which implies $v_k \to 0$.

3. Monothetic generalized convolution semigroups. In this section we shall prove that each norming sequence in (1.1) is convergent.

Given $\mu \in \mathfrak{P}$, by $\mathfrak{H}(\mu)$ we denote the closure in $\bar{\mathfrak{P}}$ of the set $\{\mu^{\circ n}: n = 1, 2, \ldots\}$. Further, by $\mathfrak{G}(\mu)$ we denote the set of all limit points in $\bar{\mathfrak{P}}$ of the sequence $\mu^{\circ n}$. Obviously,

$$\mathfrak{H}(\mu) = \mathfrak{G}(\mu) \cup \{\mu^{\circ n}: n = 1, 2, \ldots\} \tag{3.1}$$

and both sets $\mathfrak{H}(\mu)$ and $\mathfrak{G}(\mu)$ are compact.

LEMMA 3.1. *For every $\mu \in \mathfrak{P}$ the inclusion $\mathfrak{H}(\mu) \cap \mathfrak{P}_\infty \subset \{\delta_\infty\}$ is true.*

Proof. Suppose that $\nu \in \mathfrak{H}(\mu) \cap \mathfrak{P}_\infty$. Then, by (3.1), $\nu \in \mathfrak{G}(\mu)$ and, consequently, $\mu^{\circ n_k} \to \nu$ for a certain subsequence n_k tending to ∞. Put

$$\nu = c\nu' + (1-c)\delta_\infty, \tag{3.2}$$

where $\nu' \in \mathfrak{P}$ and $0 \leqslant c < 1$. Let r be an arbitrary positive integer. The sequence $n_k - r$ $(n_k > r)$ contains a subsequence m_j such that $\mu^{\circ m_j}$ is convergent, say to ν_r, when $j \to \infty$. Since $m_j + r$ is a subsequence of n_k, we have the relation

$$\mu^{\circ(m_j+r)} \to \nu \quad \text{when } j \to \infty.$$

Consequently,

$$\mu^{\circ r} \circ \nu_r = \nu \quad (r = 1, 2, \ldots). \tag{3.3}$$

Put $\nu_r = c_r \nu'_r + (1-c_r)\delta_\infty$, where $\nu'_r \in \mathfrak{P}$ and $0 \leqslant c_r \leqslant 1$. From (3.2) and (3.3) we obtain

$$c_r \nu'_r \circ \mu^{\circ r} + (1-c_r)\delta_\infty = c\nu' + (1-c)\delta_\infty \quad (r = 1, 2, \ldots), \tag{3.4}$$

which yields $c_r = c$ $(r = 1, 2, \ldots)$. We have to prove that $c = 0$. Suppose the contrary. Then, by (3.4), $\nu'_r \circ \mu^{\circ r} = \nu'$ $(r = 1, 2, \ldots)$ and, by Corollary 2.3, all limit points of the sequence $\mu^{\circ r}$ belong to $\mathfrak{P}$. But this contradicts the assumption $\nu \in \mathfrak{P}_\infty$. Thus $c = 0$, which, by (3.2), shows that $\nu = \delta_\infty$. This completes the proof.

LEMMA 3.2. *The set $\mathfrak{H}(\mu)$ is a monothetic compact semigroup under the operation $\circ$.*

Proof. It is enough to prove that the set $\mathfrak{H}(\mu)$ is closed under the operation $\circ$ and the operation in question is continuous in both variables on $\mathfrak{H}(\mu)$. Let $\nu, \lambda \in \mathfrak{H}(\mu)$. Then $\mu^{\circ n_k} \to \nu$ and $\mu^{\circ m_k} \to \lambda$ for some sequences of positive integers $n_1 \leqslant n_2 \leqslant \ldots$ and $m_1 \leqslant m_2 \leqslant \ldots$, respectively. If $(\nu, \lambda) \notin \mathfrak{P}_\infty \times \mathfrak{P}_\infty$, then, by Proposition 2.4,

$$\mu^{\circ(n_k+m_k)} \to \nu \circ \lambda,$$

which yields $\nu \circ \lambda \in \mathfrak{H}(\mu)$. In the case $(\nu, \lambda) \in \mathfrak{P}_\infty \times \mathfrak{P}_\infty$ we have, by Lemma 3.1, $\nu = \lambda = \delta_\infty$, which yields $\nu \circ \lambda = \delta_\infty$ and, consequently, $\nu \circ \lambda \in \mathfrak{H}(\mu)$. Thus $\mathfrak{H}(\mu)$ is closed under the operation $\circ$. To prove the continuity of $\circ$ in both variables on $\mathfrak{H}(\mu)$ it suffices, by Proposition 2.4 and Lemma 3.1, to prove it at the point $(\delta_\infty, \delta_\infty)$ provided $\delta_\infty \in \mathfrak{H}(\mu)$. Suppose that $\nu_n, \lambda_n \in \mathfrak{H}(\mu)$ and $\nu_n \to \delta_\infty$, $\lambda_n \to \delta_\infty$. Then, by Proposition 2.5, all limit points of $\nu_n \circ \lambda_n$ belong to $\mathfrak{H}(\mu) \cap \mathfrak{P}_\infty$ and, consequently, by Lemma 3.1, are equal to δ_∞, which completes the proof.

Applying the Numakura Theorem ([2], p. 109) we get the following statement:

COROLLARY 3.1. *The set $\mathfrak{G}(\mu)$ is a compact group under the operation $\circ$ and a minimal ideal of $\mathfrak{H}(\mu)$. Moreover, $\mathfrak{H}(\mu)$ contains exactly one idempotent, namely the unit of $\mathfrak{G}(\mu)$.*

Since δ_∞ is an idempotent, we conclude, by virtue of Lemma 3.1, that $\mathfrak{G}(\mu) = \{\delta_\infty\}$ if $\mathfrak{G}(\mu) \cap \mathfrak{P}_\infty \neq \emptyset$. This yields

COROLLARY 3.2. *For every $\mu \in \mathfrak{P}$ we have either $\mathfrak{G}(\mu) = \{\delta_\infty\}$ or $\mathfrak{G}(\mu) \subset \mathfrak{P}$.*

Suppose we have a norming sequence c_n of positive numbers such that

$$T_{c_n} \mu^{\circ n} \to \varrho, \tag{3.5}$$

where $\varrho \neq \delta_0$ and $\varrho \in \mathfrak{P}$. Denote by C the set of all its limit points in $\bar{R}_+$. Of course, C is compact.

LEMMA 3.3. *The set C is bounded.*

Proof. Contrary to this suppose $c_{n_k} \to \infty$ for a certain subsequence n_k. We may assume without loss of generality that $\mu^{\circ n_k}$ is convergent in $\bar{\mathfrak{P}}$, say to ν. By Corollary 2.4, $\nu \neq \delta_0$ and, consequently, by Proposition 2.2, all limit points of $T_{c_{n_k}} \mu^{\circ n_k}$ belong to $\mathfrak{P}_\infty$. But, by (3.5), the measure ϱ is a limit point of $T_{c_{n_k}} \mu^{\circ n_k}$ and $\varrho \in \mathfrak{P}$, which yields a contradiction. The lemma is thus proved.

LEMMA 3.4. *If $\mathfrak{G}(\mu) \subset \mathfrak{P}$, then $0 \notin C$ and*

$$G(\mu) \subset \{T_{c^{-1}} \varrho \colon c \in C\}.$$

Proof. Suppose that $0 \in C$ and $c_{n_k} \to 0$. Passing to a subsequence if necessary, we may assume that $\mu^{\circ n_k}$ is convergent, say to ν. Since $\nu \in \mathfrak{G}(\mu)$, we have $\nu \in \mathfrak{P}$. Then

$$\varrho = \lim_{t \to \infty} T_{c_{n_k}} \mu^{\circ n_k} = \delta_0,$$

which contradicts the assumption $\varrho \neq \delta_0$. Thus $0 \notin C$. Suppose now that $\lambda \in \mathfrak{G}(\mu)$ and $\mu^{\circ m_k} \to \lambda$. Taking into account Lemma 3.3, we may also assume without loss of generality that the sequence c_{m_k} converges to a positive number c. Then

$$\varrho = \lim_{k \to \infty} T_{c_{m_k}} \mu^{\circ m_k} = T_c \lambda,$$

which yields $\lambda = T_{c^{-1}} \varrho$. The lemma is thus proved.

LEMMA 3.5. *For every $\mu \in \mathfrak{P}$ we have the inclusion*

$$\{T_{c^{-1}} \varrho \colon c \in C,\ c > 0\} \subset \mathfrak{G}(\mu).$$

Proof. Suppose that $c_{n_k} \to c > 0$. Then

$$\mu^{\circ n_k} = T_{c_{n_k}^{-1}} (T_{c_{n_k}} \mu^{\circ n_k}) \to T_{c^{-1}} \varrho,$$

which yields $T_{c^{-1}} \varrho \in \mathfrak{G}(\mu)$.

From Corollary 3.2 and Lemmas 3.4 and 3.5 we get the following statements:

COROLLARY 3.3. $\mathfrak{G}(\mu) = \{\delta_\infty\}$ *if and only if* $C = \{0\}$.

COROLLARY 3.4. *If $\mathfrak{G}(\mu) \subset \mathfrak{P}$, then $0 \notin C$ and*

$$\mathfrak{G}(\mu) = \{T_{c^{-1}} \varrho \colon c \in C\}.$$

Corollary 3.4 shows that in the case $\mathfrak{G}(\mu) \subset \mathfrak{P}$ the unit λ of the group $\mathfrak{G}(\mu)$ is of the form $\lambda = T_{c^{-1}} \varrho$ for a certain positive number c. Hence ϱ is also an idempotent and, consequently, all elements of $\mathfrak{G}(\mu)$ are idempotents. Applying Corollary 3.1 we get the following statement:

COROLLARY 3.5. *If $\mathfrak{G}(\mu) \subset \mathfrak{P}$, then $\mathfrak{G}(\mu)$ and, consequently, C are one-point sets.*

Combining Corollaries 3.2–3.5 we conclude that the norming sequence c_n is always convergent to a non-negative number. Thus taking $\mu = \delta_1$ we have, by axiom (1.1), the following

THEOREM 3.1. *Each generalized convolution algebra $(\mathfrak{P}, \circ)$ is either quasi-regular or $\delta_1^{\circ n} \to \gamma$, where γ is an idempotent in $\mathfrak{P}$ different from δ_0.*

4. Idempotents. Let $\mathfrak{J}$ and $\bar{\mathfrak{J}}$ be the sets of all idempotents in the algebras $(\mathfrak{P}, \circ)$ and $(\bar{\mathfrak{P}}, \circ)$, respectively. Both sets $\mathfrak{J}$ and $\bar{\mathfrak{J}}$ are closed semigroups under the operation $\circ$ invariant under maps T_a $(a > 0)$.

LEMMA 4.1. $\bar{\mathfrak{J}} \cap \mathfrak{P}_\infty = \{\delta_\infty\}$.

Proof. The relation $\delta_\infty \in \bar{\mathfrak{J}} \cap \mathfrak{P}_\infty$ is obvious. Suppose that $\lambda \in \bar{\mathfrak{J}} \cap \mathfrak{P}_\infty$. Then $\lambda = a\lambda' + (1-a)\delta_\infty$, where $\lambda' \in \mathfrak{P}$ and $0 \leqslant a < 1$. Since

$$\lambda \circ \lambda = a^2 \lambda' \circ \lambda + (1-a^2)\delta_\infty,$$

we conclude that $a = 0$ and, consequently $\lambda = \delta_\infty$, which completes the proof.

COROLLARY 4.1. *If* $\lambda \in \mathfrak{I}$ *and* $\lambda(\{0\}) > 0$, *then* $\lambda = \delta_0$.

Proof. Setting $a = \lambda(\{0\}) > 0$, we have the formula

$$\lambda = a\delta_0 + (1-a)\lambda', \quad \text{where } \lambda' \in \bar{\mathfrak{P}} \text{ and } \lambda'(\{0\}) = 0.$$

Since $T_c\lambda = a\delta_0 + (1-a)T_c\lambda'$, we conclude, by Proposition 2.1, that

$$T_c\lambda \to a\delta_0 + (1-a)\delta_\infty \quad \text{when } c \to \infty.$$

Thus $a\delta_0 + (1-a)\delta_\infty \in \mathfrak{I}$. Applying Lemma 4.1, we obtain $a = 1$, which completes the proof.

LEMMA 4.2. *If* $\mu \in \mathfrak{P}$ *and* $T_{a_n}\mu^{\circ n} \to \delta_0$, *then*

$$\mu^n([0, a_n^{-1}b]) \to 1 \quad \textit{for every } b > 0.$$

Proof. Consider the countable product

$$R_+^* = R_+ \times R_+ \times \ldots$$

with the Tihonov topology and the product measure

$$\mu^* = \mu \times \mu \times \ldots$$

on Borel subsets of R_+^*. Using the notation

$$y = (y_1, y_2, \ldots) \in R_+^*$$

we have, by (2.13),

$$T_{a_n}\mu^{\circ n} = \int_{R_+} \ldots \int_{R_+} \mathop{\bigcirc}_{j=1}^{n} \delta_{a_n y_j} \mu(dy_1) \ldots \mu(dy_n)$$

$$= \int_{R_+} \mathop{\bigcirc}_{j=1}^{n} \delta_{a_n y_j} \mu^*(dy) \quad (n = 1, 2, \ldots).$$

Suppose that $T_{a_n}\mu^{\circ n} \to \delta_0$. Then, by Lemma 2.1, each monotone increasing sequence of positive integers contains a subsequence $n_1 < n_2 < \ldots$ such that

$$\mathop{\bigcirc}_{j=1}^{n_k} \delta_{a_{n_k} y_j} \to \delta_0 \quad \text{for all } y \in Y_0,$$

where Y_0 is a Borel subset of R_+^* with $\mu^*(Y_0) = 1$. As a consequence of Corollary 2.5 we have the relation

$$\max\{a_{n_k} y_j \colon j = 1, 2, \ldots, n_k\} \to 0$$

for all $y \in Y_0$. Thus

$$\mu^*\{y \colon \max\{a_{n_k} y_j \colon j = 1, 2, \ldots, n_k\} \leqslant b\} \to 1$$

for every $b > 0$. The probability which figures on the left-hand side of the above formula is equal to $\mu^{n_k}([0, a_{n_k}^{-1} b])$. This shows that every subsequence of the sequence $\mu^n([0, a_n^{-1} b])$ contains a subsequence tending to 1, which yields the assertion of the lemma.

LEMMA 4.3. *Each idempotent from $\mathfrak{J}$ has a bounded support.*

Proof. It is enough to consider idempotents ν different from δ_0. Let k be a positive integer satisfying the inequality $1/k < 1-\nu(\{0\})$. Then for $n \geq k$ the sets

$$B_n = \{b\colon\ \nu([0, b]) \leq 1-1/n\}$$

are non-degenerate intervals. Setting $b_n = \sup B_n$, we have $0 < b_n \leq b_{n+1} < \infty$ $(n \geq k)$ and

$$\nu([0, b_n+\varepsilon]) > 1-1/n \qquad (n \geq k) \tag{4.1}$$

for every $\varepsilon > 0$. We shall prove that the sequence b_n is bounded. Contrary to this suppose that $b_n \to \infty$ and put $a_n = 2/b_n$ $(n \geq k)$. Then $a_n \to 0$ and, consequently,

$$T_{a_n} \nu^{\circ n} = T_{a_n} \nu \to \delta_0 .$$

Applying Lemma 4.2 we have

$$\nu^n([0, b_n/2]) \to 1. \tag{4.2}$$

On the other hand, $\nu([0, b_n/2]) \leq 1-1/n$ $(n \geq k)$, which yields

$$\nu^n([0, b_n/2]) \leq (1-1/n)^n \qquad (n \geq k).$$

The right-hand side of the last inequality tends to e^{-1} when $n \to \infty$, which contradicts (4.2). Thus the sequence b_n is bounded and, consequently, has a finite limit, say b. By (4.1) we conclude that $\nu([0, b+1]) = 1$, which shows that ν has a bounded support. The lemma is thus proved.

Given $\mu \in \mathfrak{P}$, we denote by $m(\mu)$ any median of μ. It is clear that $m(T_c \mu) = cm(\mu)$ $(c > 0)$ and the relations $\mu_n \to \mu$ and $m(\mu_n) \to m$ imply $m(\mu) = m$.

LEMMA 4.4. *If $\lambda_n \in \mathfrak{J}$ and $m(\lambda_n) \to 0$, then $\lambda_n \to \delta_0$.*

Proof. Let λ be a limit point of the sequence λ_n. Then $\lambda \in \mathfrak{J}$, $m(\lambda) = 0$ and, consequently, $\lambda(\{0\}) > 0$. Applying Corollary 4.1 we get $\lambda = \delta_0$, which completes the proof.

An idempotent λ from $\mathfrak{J}$ is said to be *completely stable* if for any pair $a, b \in R_+$ the formula

$$T_a \lambda \circ T_b \lambda = T_{\max(a,b)} \lambda$$

holds.

LEMMA 4.5. *If $\mathfrak{J} \neq \{\delta_0\}$, then $\mathfrak{J}$ contains a completely stable idempotent different from δ_0.*

Proof. Suppose that $\mu \in \mathfrak{J}$ and $\mu \neq \delta_0$. Then, by Lemma 4.4, $m(\mu) > 0$. Passing to a measure $T_c \mu$ $(c > 0)$ if necessary, we may assume that $m(\mu) = 1$. Put $x_1 = 1$ and let $x_2, x_3, \ldots$ $(0 < x_j < 1,\ j = 2, 3, \ldots)$ be a sequence dense in $[0, 1]$. First we shall construct inductively a sequence ν_r of idempotents from $\mathfrak{J}$ satisfying the conditions

$$m(\nu_r) = 1, \tag{4.3}$$

$$T_{x_j} \nu_r \circ \nu_r = \nu_r \quad (j = 1, 2, \ldots, r). \tag{4.4}$$

We define ν_1 as μ. Suppose that for some r the measure ν_r has been constructed to satisfy conditions (4.3) and (4.4). We shall construct ν_{r+1}. Put

$$\mu_n = \nu_r \circ \mathop{\bigcirc}_{j=1}^{n} T_{x^j_{r+1}} \nu_r \quad (n = 1, 2, \ldots) \tag{4.5}$$

and introduce the notation $m(\mu_n) = m_n$. Since, by (4.3), $\nu_r \neq \delta_0$, we conclude, by Corollary 2.4, that δ_0 is not a limit point of the sequence μ_n. Moreover, $\mu_n \in I$, which, by Lemma 4.4, yields

$$\varliminf_{n \to \infty} m_n > 0. \tag{4.6}$$

Put $\varrho_n = T_{m_n^{-1}} \mu_n$ $(n = 1, 2, \ldots)$. Then $\varrho_n \in \mathfrak{J}$ and

$$m(\varrho_n) = 1 \quad (n = 1, 2, \ldots). \tag{4.7}$$

Further, taking into account (4.4) and (4.5), we have

$$T_{x_j} \varrho_n \circ \varrho_n = \varrho_n \quad (j = 1, 2, \ldots, r) \tag{4.8}$$

and

$$T_{x_{r+1}} \varrho_n \circ \varrho_n = \varrho_n \circ \omega_n, \tag{4.9}$$

where $\omega_n = T_{m_n^{-1} x_{r+1}^{n+1}} \nu_r$. Since $0 < x_{r+1} < 1$, we have, by (4.6),

$$\omega_n \to \delta_0. \tag{4.10}$$

We define ν_{r+1} as an arbitrary limit point of the sequence ϱ_n. By (4.7) we have $m(\nu_{r+1}) = 1$, which shows, according to Lemma 4.1, that $\nu_{r+1} \in \mathfrak{J}$. Moreover, by Theorem 2.1 and formulas (4.8)–(4.10) we get the equation

$$T_{x_j} \nu_{r+1} \circ \nu_{r+1} = \nu_{r+1} \quad (j = 1, 2, \ldots, r+1).$$

This completes the induction. Let λ be a limit point of the sequence ν_r. By (4.3) we have $m(\lambda) = 1$, which, by Lemma 4.1, yields $\lambda \in \mathfrak{J}$. Moreover, $\lambda \neq \delta_0$ and, by (4.4), we have $T_{x_j} \lambda \circ \lambda = \lambda$ $(j = 1, 2, \ldots)$. Since the sequence x_j is dense in $[0, 1]$, the last equation implies $T_x \lambda \circ \lambda = \lambda$ $(0 \leqslant x \leqslant 1)$. Combining this equation with the distributivity of $\circ$ with respect to all maps T_c $(c > 0)$

we infer that the idempotent λ is completely stable, which completes the proof.

LEMMA 4.6. *If a generalized convolution algebra* $(\mathfrak{P}, \circ)$ *contains a completely stable idempotent different from* δ_0, *then* $\circ = \square$.

Proof. Let λ be a completely stable idempotent in $\mathfrak{I}$ different from δ_0. By Lemma 4.3 the support $N(\lambda)$ is bounded. Passing to $T_c\lambda$ for a certain $c > 0$ if necessary, we may assume without loss of generality that

$$N(\lambda) \subset [0, 1] \quad \text{and} \quad 1 \in N(\lambda). \tag{4.11}$$

By (2.13), for any $a, b \in R_+$ we have the formula

$$T_{\max(a,b)}\lambda = T_a\lambda \circ T_b\lambda = \int_{R_+}\int_{R_+} \delta_{at} \circ \delta_{bu}\, \lambda(dt)\,\lambda(du),$$

which, by Proposition 2.3 and (4.11), yields

$$N(\delta_{at} \circ \delta_{bu}) \subset N(T_{\max(a,b)}\lambda) \subset [0, \max(a, b)]$$

for all $t, u \in N(\lambda)$. Taking, by (4.11), $u = t = 1$, we get

$$N(\delta_a \circ \delta_b) \subset [0, \max(a, b)] \quad (a, b \in R_+).$$

Then it follows from (2.13) that for every $c > 0$

$$N(\mu \circ \nu) \subset [0, c] \quad \text{if } N(\mu) \cup N(\nu) \subset [0, c]. \tag{4.12}$$

Taking $c < 1$ we have, by (4.11),

$$a(c) = \lambda((c, 1]) > 0. \tag{4.13}$$

Put

$$\mu_c(E) = a^{-1}(c)\lambda(E \cap (c, 1]), \tag{4.14}$$

$\nu_c(E) = (1-a(c))^{-1}\lambda(E \cap [0, c])$ if $a(c) < 1$ and $\nu_c = \delta_0$ if $a(c) = 1$. Then, by (4.11) and (4.12),

$$N(T_x\lambda \circ \nu_c) \subset [0, c] \quad (0 \leqslant x \leqslant c),$$

which implies

$$(T_x\lambda \circ \nu_c)((c, 1]) = 0 \quad (0 \leqslant x \leqslant c). \tag{4.15}$$

Moreover, $\lambda = a(c)\mu_c + (1-a(c))\nu_c$ and

$$\lambda = T_x\lambda \circ \lambda = a(c)\mu_c \circ T_x\lambda + (1-a(c))T_x\lambda \circ \nu_c \quad (0 \leqslant x \leqslant c).$$

Taking into account (4.15) we conclude that

$$\lambda((c, 1]) = a(c)(T_x\lambda \circ \mu_c)((c, 1]) \quad (0 \leqslant x \leqslant c),$$

which together with (4.13) yields

$$(T_x\lambda \circ \mu_c)((c, 1]) = 1 \quad (0 \leqslant x \leqslant c).$$

Consequently, $T_x\lambda \circ \mu_c \to \delta_1$ when $c \to 1$ $(0 \leqslant x < 1)$. Since, by (4.14), $\mu_c \to \delta_1$ when $c \to 1$, we have the equation $T_x\lambda \circ \delta_1 = \delta_1$ $(0 \leqslant x < 1)$. Then it follows from (2.13) that

$$\delta_1 = T_x\lambda \circ \delta_1 = \int_{R_+} \delta_{xt} \circ \delta_1 \, \lambda(dt) \qquad (0 \leqslant x < 1),$$

which, by Proposition 2.3, yields $N(\delta_{xt} \circ \delta_1) \subset \{1\}$ or, equivalently, $\delta_{xt} \circ \delta_1 = \delta_1$ for $t \in N(\lambda)$ and $0 \leqslant x \leqslant 1$. Taking, according to (4.11), $t = 1$ we get $\delta_x \circ \delta_1 = \delta_1$ if $0 \leqslant x \leqslant 1$, which yields

$$\delta_a \circ \delta_b = \delta_{\max(a,b)} \quad \text{for all } a,\, b \in R_+.$$

Thus, by (1.2), $\delta_a \circ \delta_b = \delta_a \square \delta_b$ $(a, b \in R_+)$ which, by (2.13), gives the equation $\mu \circ \nu = \mu \square \nu$ for all $\mu, \nu \in \mathfrak{P}$. Thus $\circ = \square$, which completes the proof.

As a consequence of Theorem 3.1 and Lemmas 4.5 and 4.6 we get the following statements:

THEOREM 4.1. *The generalized convolution algebra* $(\mathfrak{P}, \square)$ *is the only non-quasi-regular algebra.*

THEOREM 4.2. *A generalized convolution algebra* $(\mathfrak{P}, \circ)$ *is quasi-regular if and only if* δ_0 *is the only idempotent in* $(\mathfrak{P}, \circ)$.

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Reçu par la Rédaction le 20.4.1984