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COMPLETELY STABLE MEASURES ON HILBERT SPACES

BY

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This study continues the investigations of completely stable probability measures started by Parthasarathy in [3]. He proved that every non-degenerate completely stable probability measure on the at least two-dimensional Euclidean space is Gaussian. Our aim is to prove that in the infinite-dimensional case there exist non-Gaussian completely stable measures. Moreover, we shall give a characterization of completely stable Gaussian measures in terms of proper values of their covariance operators.

Let H be a real separable Hilbert space with the inner product $(\cdot\,,\cdot)$ and the norm $\|\cdot\|$. By a *probability measure* on H we shall understand a countably additive non-negative set function μ on the class of Borel subsets of H with the property $\mu(H) = 1$. For two probability measures μ and ν on H, we shall denote by $\mu*\nu$ the convolution of μ and ν. Further, by $B(H)$ we shall denote the set of all continuous linear operators on H. Given a probability measure μ on H and on operator A from $B(H)$, by $A\mu$ we shall denote the probability measure defined by the formula $A\mu(E) = \mu\big(A^{-1}(E)\big)$ for Borel subsets E of H. In other words, if μ is the probability distribution of an H-valued random variable ζ, then $A\mu$ is the probability distribution of $A\zeta$. In what follows δ_x will denote a degenerate probability measure concentrated at the point x $(x \in H)$.

A probability measure μ on H is said to be *completely stable* if for any pair $A, B \in B(H)$ there exist $C \in B(H)$ and $x \in H$ such that

$$A\mu*B\mu = C\mu*\delta_x. \tag{1}$$

Suppose that S is the covariance operator of a completely stable probability measure μ. Since ASA^* is the covariance operator of $A\mu$, we infer that for any pair $A, B \in B(H)$ there exists an operator $C \in B(H)$ such that

$$ASA^*+BSB^* = CSC^*. \tag{2}$$

LEMMA 1. *Non-degenerate completely stable measures on infinite-dimensional spaces H are not concentrated on finite-dimensional hyperplanes.*

Proof. Suppose that μ is completely stable and concentrated on a finite-dimensional hyperplane of H. Let k be the least integer for which there exist a k-dimensional subspace H_0 of H and an element x_0 of H such that μ is concentrated on the hyperplane H_0+x_0. Since H is infinite dimensional, we can find a unitary operator U on H with the property $UH_0 \subset H_0^{\perp}$. Let C be an operator satisfying the condition $\mu * U\mu = C\mu * \delta_x$ for a certain element $x \in H$. Then $CH_0 = H_0 \oplus UH_0$ and, consequently, $k \geqslant \dim CH_0 = 2k$ which yields $k = 0$. Thus μ is degenerate which completes the proof.

LEMMA 2. *Let $a_1 \geqslant a_2 \geqslant \ldots$ and $b_1 \geqslant b_2 \geqslant \ldots$ be the sequences of eigenvalues of covariance operators S and CSC^* $(C \in B(H))$, respectively. Then the inequality*

$$b_n \leqslant \|C\|^2 a_n \quad (n = 1, 2, \ldots)$$

holds.

Proof. Taking into account the formula for eigenvalues of non-negative operators ([1], Theorem 3, Chapter 10.4) we have

$$b_1 = \max\left\{\frac{(CSC^* x, x)}{\|x\|^2} : x \in H\right\},$$

$$b_{n+1} = \min_{v_1, v_2, \ldots, v_n \in H} \max\left\{\frac{(CSC^* x, x)}{\|x\|^2} : x \in H, (x, y_j) = 0, j = 1, 2, \ldots, n\right\}.$$

Consequently,

$$b_1 \leqslant \|C\|^2 \max\left\{\frac{(SC^* x, C^* x)}{\|C^* x\|^2} : x \in H\right\} \leqslant \|C\|^2 a_1$$

and

$$b_{n+1} \leqslant \|G\|^2 \min_{u_1, u_2, \ldots, u_n \in H} \max\left\{\frac{(SC^* x, C^* x)}{\|C^* x\|^2} : x \in H, (x, Cu_j) = 0, \right.$$

$$\left. j = 1, 2, \ldots, n\right\}$$

$$\leqslant \|G\|^2 \min_{u_1, u_2, \ldots, u_n \in H} \max\left\{\frac{(Sy, y)}{\|y\|^2} : y \in H, (y, u_j) = 0, j = 1, 2, \ldots, n\right\}$$

$$\leqslant \|G\|^2 a_{n+1},$$

which completes the proof.

Suppose we have a sequence $a_1 \geqslant a_2 \geqslant \ldots$ of positive numbers such that the sequence

(3) a_n / a_{2n} $(n = 1, 2, \ldots)$ is unbounded.

We note that for every pair n, m of positive integers there exists an index $k(n,m)$ satisfying the condition $a_n \leqslant ma_j$ for $j \leqslant k(n,m)$ and $a_n > ma_j$ otherwise. Evidently,

$$k(n+1,m) \geqslant k(n,m) \quad (n,m=1,2,\ldots). \tag{4}$$

Now we prove two lemmas concerning the sequence $a_1, a_2, \ldots$ and the indices $k(n,m)$.

LEMMA 3. *There exists a subsequence $n_1 < n_2 < \ldots$ of positive integers such that*

$$n_m > l(m-1) \quad (m=1,2,\ldots), \tag{5}$$

$$l(m) > k(n_m, m) \quad (m=1,2,\ldots), \tag{6}$$

where $l(0)=0$ and

$$l(m) = 2(-1)^m \sum_{j=1}^{m} (-1)^j n_j \quad (m=1,2,\ldots).$$

Proof. It is evident that the inequality $2n \leqslant k(m,1)$ for all n would imply $a_n \leqslant a_{2n}$ $(n=1,2,\ldots)$. But this contradicts (3). Consequently, we can choose an index n_1 with the property $l(1) = 2n_1 > k(n_1,1)$. Proving by induction we assume that the indices $n_1, n_2, \ldots, n_{m-1}$ $(m \geqslant 2)$ are chosen and fulfil (5) and (6). We note that the inequality

$$2n - l(m-1) \leqslant k(n,m)$$

for all $n > l(m-1)$ would imply

$$a_n \leqslant ma_{2n-l(m-1)} \quad (n > l(m-1)). \tag{7}$$

But this inequality yields

$$a_{2n-l(m-1)} \leqslant ma_{4n-3l(m-1)} \quad (n > l(m-1)) \tag{8}$$

because $2n - l(m-1) > l(m-1)$. Since for sufficiently large n the inequality $4n - 3l(m-1) > 2n$ holds, we have, by virtue of (4) and (8),

$$a_{2n-l(m-1)} \leqslant ma_{2n}$$

which, by (7), yields

$$a_n \leqslant m^2 a_{2n}$$

for sufficiently large n. But this contradicts assumption (3). Consequently, there exists an index n_m with properties (5) and

$$l(m) = 2n_m - l(m-1) > k(n_m, m).$$

The lemma is thus proved.

We define the permutation q of positive integers by setting

$$q(n_m+j) = l(m-1)+j \quad (j = 1, 2, \ldots, n_m-l(m-1)), \tag{9}$$

$$q\big(l(m-1)+j\big) = n_m+j \quad (j = 1, 2, \ldots, n_m-l(m-1)). \tag{10}$$

LEMMA 4. *For every permutation p of positive integers the sequence $(a_n+a_{q(n)})/a_{p(n)}$ $(n = 1, 2, \ldots)$ is unbounded.*

Proof. Contrary to this suppose that the sequence $(a_n+a_{q(n)})/a_{p(n}$ $(n = 1, 2, \ldots)$ is bounded for a permutation p. There exists then a positive integer m such that

$$a_n \leqslant m a_{p(n)} \tag{11}$$

and

$$a_{q(n)} \leqslant m a_{p(n)} \tag{12}$$

for all indices n. By (4) and (11) the inequality

$$p(j) \leqslant k(n, m) \quad (j = 1, 2, \ldots, n) \tag{13}$$

holds. Moreover, (12) yields

$$p(r) \leqslant k\big(q(r), m\big) \quad (r = 1, 2, \ldots). \tag{14}$$

Setting $r = n_m+j$ $(j = 1, 2, \ldots, n_m-l(m-1))$, we have, by (9),

$$q(r) = l(m-1)+j \leqslant n_m$$

which, by virtue of (4) and (14), implies

$$p(n_m+j) \leqslant k(n_m, m) \quad (j = 1, 2, \ldots, n_m-l(m-1)).$$

On the other hand, setting $n = n_m$ into (13), we get

$$p(j) \leqslant k(n_m, m) \quad (j = 1, 2, \ldots, n_m).$$

Thus we have proved that the permutation p transforms at least $2n_m-l(m-1)$ indices into positive integers less than or equal to $k(n_m, m)$. Since $l(m) = 2n_m-l(m-1)$, we have the inequality $l(m) \leqslant k(n_m, m)$ which contradicts (6). The lemma is thus proved.

The following statement is a direct consequence of Lemma 4.

COROLLARY 1. *Let $b_1 \geqslant b_2 \geqslant \ldots$ be a rearrangement of the terms of the sequence $a_n+a_{q(n)}$ $(n = 1, 2, \ldots)$. Then the sequence b_n/a_n $(n = 1, 2, \ldots)$ is unbounded.*

It is very easy to verify that every probability measure the covariance operator of which has only a finite number of positive eigenvalues is concentrated on a finite-dimensional hyperplane of H.

THEOREM 1. *Suppose that*

$$\int_H \|x\|^2 \mu(dx) < \infty$$

and that μ is not concentrated on any finite-dimensional hyperplane of H. Let $a_1 \geqslant a_2 \geqslant \ldots$ be the sequence of all positive eigenvalues of the covariance operator of μ. If the sequence a_n/a_{2n} $(n = 1, 2, \ldots)$ is unbounded, then μ is not completely stable.

Proof. Let S denote the covariance operator of μ and let $e_1, e_2, \ldots$ be an orthonormal sequence of eigenvectors of S corresponding to the eigenvalues $a_1, a_2, \ldots$, respectively. Taking the permutation q defined by (9) and (10) we define a unitary operator U by setting $U^* e_n = e_{q(n)}$ $(n = 1, 2, \ldots)$ and assuming U to be the identity operator on the orthogonal complement of the subspace induced by $e_1, e_2, \ldots$ It is easy to verify that the covariance operator $S + USU^*$ of the probability measure $\mu * U\mu$ has the eigenvectors e_n corresponding to the eigenvalues $a_n + a_{q(n)}$ $(n = 1, 2, \ldots)$, respectively. Contrary to the assertion of the theorem let us suppose that μ is completely stable. Then there exist an operator $C \in B(H)$ and an element $x \in H$ such that $\mu * U\mu = C\mu * \delta_x$. Moreover, by (2), $CSC^* = S + USU^*$. Thus the sequence $b_1 \geqslant b_2 \geqslant \ldots$ of positive eigenvalues of CSC^* is a rearrangement of the sequence $a_n + a_{q(n)}$ $(n = 1, 2, \ldots)$. By Lemma 2 the sequence b_n/a_n $(n = 1, 2, \ldots)$ is bounded which contradicts Corollary 1. This completes the proof.

THEOREM 2. *Suppose that the Hilbert space H is infinite dimensional and $e_1, e_2, \ldots$ is an orthonormal sequence in H. Further, suppose that $\xi_1, \xi_2, \ldots$ is a sequence of independent identically distributed real-valued random variables, $c_1, c_2, \ldots$ is a sequence of positive numbers, c_n/c_{2n} $(n = 1, 2, \ldots)$ is bounded, and the series $\sum_{n=1}^{\infty} c_n^2 \xi_n^2$ converges almost surely. Then the probability distribution of the H-valued random variable*

$$\zeta = \sum_{n=1}^{\infty} c_n \xi_n e_n \tag{15}$$

is completely stable.

Proof. The boundedness of c_n/c_{2n} $(n = 1, 2, \ldots)$ enables us to define two auxiliary linear operators T and V on H by assuming

$$Te_{2n} = \frac{c_n}{c_{2n}} e_n, \quad Te_{2n-1} = 0 \quad (n = 1, 2, \ldots),$$

$$Ve_{2n-1} = \frac{c_n}{c_{2n-1}} e_n, \quad Ve_{2n} = 0 \quad (n = 1, 2, \ldots)$$

and $T = V = 0$ on the orthogonal complement of the subspace generated by $e_1, e_2, \ldots$ Put

$$\zeta_1 = \sum_{n=1}^{\infty} c_n \xi_{2n} e_n \quad \text{and} \quad \zeta_2 = \sum_{n=1}^{\infty} c_n \xi_{2n-1} e_n.$$

The random variables ζ_1 and ζ_2 are independent and have the same probability distribution as ζ. Moreover, for any pair A, B from $B(H)$, setting $C = AT+BV$, we have the formula

$$C\zeta = A\zeta_1+B\zeta_2$$

which shows that ζ has the completely stable probability distribution. The theorem is thus proved.

Example 1. Every sequence $\xi_1, \xi_2, \ldots$ of independent identically distributed random variables with a finite variance and the sequence $c_n = 1/n$ $(n = 1, 2, \ldots)$ fulfil the conditions of Theorem 2. In particular, if we choose $\xi_1, \xi_2, \ldots$ to be not infinitely divisible, then also the random variable (15) is not infinitely divisible.

Example 2. Taking p-stable random variables $(0 < p < 2)$ as the random variables $\xi_1, \xi_2, \ldots$ and $c_n = 1/n^r$ $(n = 1, 2, \ldots)$, where $rp > 1$, we infer that the random variable (15) has the infinite p-th moment, i.e. $\mathrm{E}\,\|\zeta\|^p = \infty$.

THEOREM 3. *Suppose that the Hilbert space H is infinite dimensional. Then a non-degenerate Gaussian measure on H is completely stable if and only if its covariance operator has infinitely many positive eigenvalues $a_1 \geqslant a_2 \geqslant \ldots$ and the sequence a_n/a_{2n} $(n = 1, 2, \ldots)$ is bounded.*

Proof. First let us assume that a non-degenerate Gaussian measure is completely stable. Then, by Lemma 1, it is not concentrated on any finite-dimensional hyperplane of H. Consequently, its covariance operator has infinitely many positive eigenvalues $a_1 \geqslant a_2 \geqslant \ldots$ Taking into account Theorem 1, we infer that the sequence a_n/a_{2n} $(n = 1, 2, \ldots)$ is bounded.

Conversely, suppose that ζ is an H-valued Gaussian random variable and that its covariance operator has infinitely many positive eigenvalues $a_1 \geqslant a_2 \geqslant \ldots$ such that the sequence a_n/a_{2n} $(n = 1, 2, \ldots)$ is bounded. Then ζ can be written as

$$\zeta = x_0+\sum_{n=1}^{\infty}\sqrt{a_n}\,\xi_n e_n,$$

where x_0 is the mean value of ζ, $e_1, e_2, \ldots$ is an orthonormal sequence in H, and $\xi_1, \xi_2, \ldots$ are independent standard Gaussian random variables ([2], Theorem 6.3.2). Now our statement is a direct consequence of Theorem 2.

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