COLLOQUIUM MATHEMATICUM

VOL. XL 1978 FASC. 1

ON FILLING AN IRREDUCIBLE CONTINUUM WITH ONE POINT UNION OF 2-CELLS

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In this paper, a *continuum* means a compact connected metric space.

Let $\mathscr{K}$ denote the class of all continua K such that there exists an upper semi-continuous decomposition G of an irreducible continuum M with each element G homeomorphic to K and with decomposition space M/G as an arc. It is shown [1] that the n-cell is in $\mathscr{K}$. In this paper it is shown that if K is the sum of two 2-cells joined at a single point, then K is in $\mathscr{K}$. The construction can be modified to include other continua.

THEOREM 1. *There exists a continuum M in the plane satisfying the following conditions*:

(1) *M is irreducible.*

(2) *There exists an upper semi-continuous collection G of arcs filling up M such that M/G is an arc.*

(3) *There exists a countable subcollection H of G such that*

(a) *if h is in H, then h contains an arc z_h such that each point of z_h is a separating point of M in h;*

(b) *if S denotes the set of all points P such that P is an endpoint of z_h for some h in H, then S is dense in $M \dot{-} \bigcup_{h\in H} z_h$;*

(c) *if $\varepsilon > 0$, then only finitely many members h of H have diameter $d(z_h) > \varepsilon$;*

(d) *$\bigcup_{h\in H} z_h$ contains all separating points of M;*

(e) *there exists a line l such that if h is in H, then z_h is a subset of l or does not intersect l;*

(f) *if h is in H and z_h does not intersect l, then the endpoints of h are on l, and if S' denotes the set of all endpoints of elements of H that are on l, then $\overline{S'}$ is a Cantor set on l;*

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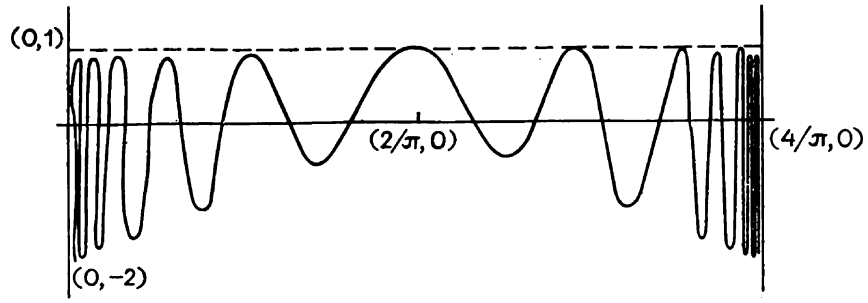
[1] See J. W. Hinrichsen, *Concerning irreducible continua of higher dimension*, Colloquium Mathematicum 28 (1973), p. 227-230.

(g) *if Z_l denotes the set of all z_h's on l, then Z_l is infinite and $\overline{M-Z_l^*}$ is a closed point set each component of which is an arc.*

Remark. The continuum M described above is a modification of the continuum described in Theorem 1 of Hinrichsen (op. cit.). Much of the notation used in the present construction is used throughout this paper.

Proof. Let

$$g_1(x) = \begin{cases} \left(\sin\frac{1}{x}\right)\left[-\frac{\pi}{2}x+\frac{3}{2}+\left(\frac{\pi}{2}x-\frac{1}{2}\right)\sin\frac{1}{x}\right] & \text{if } 0 < x \leqslant \frac{2}{\pi}, \\ \sin\frac{1}{4/\pi - x}\left\{-\frac{\pi}{2}\left(\frac{4}{\pi}-x\right)+\frac{3}{2}+\left[\frac{\pi}{2}\left(\frac{4}{\pi}-x\right)-\frac{1}{2}\right]\sin\frac{1}{4/\pi-x}\right\} & \text{if } \frac{2}{\pi} \leqslant x < \frac{4}{\pi}. \end{cases}$$



Let

$$g_2(x) = \max\{g_1(x)-1,\ -2\}, \qquad 0 < x < \frac{4}{\pi}.$$

Denote by g_1 and g_2 the graphs of the defined functions and let g_3 denote the closure of the set of all points of g_2 above the line $y = -2$. Let $g_4 = g_2 - g_3$. Let A denote the vertical interval from $(0, -2)$ to $(0, 1)$ and let B denote the vertical interval from $(4/\pi, -2)$ to $(4/\pi, 1)$. Let H_1 denote $A \cup B \cup g_1 \cup g_2$ and let I_1 denote the bounded domain of the complement of $A \cup B \cup g_1 \cup g_2$. Let a_1 denote a countable sequence of mutually exclusive arcs such that if u is in a_1, then

(1) u lies in $H_1 \cup I_1$.

(2) The endpoints of u both lie on g_1 or on a component of g_4.

(3) u intersects g_1 and g_4 and $u \cap (g_1 \cup g_4)$ is the set consisting of the endpoints of u together with an arc z_u.

(4) The diameter of each component of $u - u \cap (g_1 \cup g_4)$ is greater than 1.

(5) If $d(z_u)$ denotes the diameter of z_u for each u in a_1, then

$$\sum_{u \in a_1} d(z_u) \leqslant \frac{1}{2}.$$

(6) If S_1 denotes the set of all points P such that P is an endpoint of z_u for some u in a_1, then the limiting set of S_1 is $A \cup B$.

(7) The closure of the set of all components of g_4 which contains z_u for some u in a_1 contains the points $(0, -2)$ and $(4/\pi, -2)$.

For an element u of a_1, let D_u denote the component of $(H_1 \cup I_1) - u$ that does not contain A or B. Let L_1 denote

$$(H_1 \cup I_1) - \bigcup_{u \in a_1} D_u.$$

Let C_1 denote the family of the components of $L_1 - (a_1^* \cup A \cup B)$. Let $c_{11}, c_{12}, c_{13}, \ldots$ denote the elements of C_1. For an element c of C_1, let A_c and B_c denote the components of $\bar{c} \cap a_1^*$, and let g_{1c} and g_{4c} stand for the components of $\bar{c} \cap (g_1 \cup g_4)$. Let x and y denote the intervals $[(0, 1), (4/\pi, 1)]$ and $[(0, 2), (4/\pi, -2)]$, respectively.

Let $f_{c_{11}}, f_{c_{12}}, f_{c_{13}}, \ldots$ denote a sequence such that, for each i, $f_{c_{1i}}$ is a homeomorphism from the square disc bounded by $A \cup B \cup x \cup y$ onto $\bar{c}_{1i}$, which satisfies the following conditions:

(1) $f_{c_{1i}}(A \cup B) = A_{c_{1i}} \cup B_{c_{1i}}$;

(2) $f_{c_{1i}}(x \cup y) = g_{1c_{1i}} \cup g_{4c_{1i}}$;

(3) if u is in the sequence a_1, then the diameter of each component of $f_{c_{1i}}(u - u \cap (g_1 \cup g_2))$ is greater than 1;

(4) $f_{c_{1i}}(z_u) \subset g_4$ for each u in a_1 such that z_u is a subset of g_4, and if the endpoints of an element u in a_1 belong to a component of g_4, then the endpoints of $f_{c_{1i}}(u)$ both belong to some component of g_4;

(5) the area of

$$L_2 = A \cup B \cup a_1^* \cup \bigcup_{i>0} f_{c_{1i}}(L_1)$$

is less than one half the area of L_1;

(6) $\sum_{i>0} \sum_{u \in a_1} d[f_{c_{1i}}(z_u)] \leqslant \frac{1}{4}$.

Continuing inductively, let a_n denote the collection of arcs to which v belongs if and only if, for some element $c_{n-1,i}$ of C_{n-1}, v is $f_{c_{n-1,i}}(u)$ for some element u of a_1. Let C_n denote the family of the components of

$$L_n - (a_1^* \cup a_2^* \cup a_3^* \cup \ldots \cup a_n^* \cup A \cup B).$$

For an element c of C_n, let A_c and B_c denote the components of $\bar{c}\cap a_n^*$ and let g_{1c} and g_{4c} stand for the closures of the components of $B(\bar{c})-(A_c\cup B_c)$, where $B(\bar{c})$ is the boundary of $\bar{c}$.

Let $f_{c_{n1}}, f_{c_{n2}}, f_{c_{n3}}, \ldots$ denote a sequence such that, for each i, $f_{c_{ni}}$ is a homeomorphism from the square disc bounded by $A\cup B\cup x\cup y$ onto $\bar{c}_{ni}$, which satisfies the following conditions:

(1) $f_{c_{ni}}(A\cup B) = A_{c_{ni}}\cup B_{c_{ni}}$;

(2) $f_{c_{ni}}(x\cup y) = g_{1c_{ni}}\cup g_{4c_{ni}}$;

(3) if u is in the sequence a_1, then the diameter of each component of $f_{c_{ni}}\big(u-u\cap(g_1\cup g_4)\big)$ is greater than 1;

(4) $f_{c_{ni}}(z_u)\subset g_4$ for each u in a_1 such that z_u is a subset of g_4, and if the endpoints of an element u of a_1 belong to a component of g_4, then the endpoints of $f_{c_{ni}}(u)$ both belong to g_4;

(5) the area of

$$L_{n+1} = A\cup B\cup \bigcup_{i>0} f_{c_{ni}}(L_1)\cup a_1^*\cup a_2^*\cup a_3^*\cup \ldots \cup a_n^*$$

is less than $A(L_1)/(n+1)$, where $A(L_1)$ is the area of L_1;

(6) $$\sum_{i>0}\sum_{u\in a_1} d[f_{c_{ni}}(z_u)] \leqslant \frac{1}{2n}.$$

$L_1, L_2, L_3, \ldots$ is a monotone sequence of compact continua and the common part L of all of them is an irreducible continuum, since the set of all points of L which separate A from B in L is dense in L. The collection to which h belongs if and only if, for some u of a_1, some n, and some i, h is $f_{c_{ni}}(z_u)$, is a countable collection of mutually exclusive arcs satisfying the condition of the conclusion of Theorem 1.

Let K denote the collection to which g belongs if and only if g is a point of $(A\cup B\cup a_1^*\cup a_2^*\cup a_3^*\cup \ldots)$ or, for some component c of

$$L-\Big(A\cup B\cup \bigcup_{n=1}^{\infty} a_n^*\Big)$$

and for some horizontal line l intersecting c, g is the set of all points of c on l. K is an upper semi-continuous collection of mutually exclusive closed point sets filling up L. Let M denote L/K. Let G denote the collection to which g belongs if and only if g is A, B, or an element of a_n for some n, or g is a component of

$$M-\Big(A\cup B\cup \bigcup_{n=1}^{\infty} a_n^*\Big).$$

M is an irreducible continuum from A to B, and M/G is an arc. Furthermore, each element of G is an arc. Also, M is chainable and, therefore, embeddable in the plane. It can be seen that M satisfies all the conditions of the conclusion of Theorem 1 by letting

$$H = \bigcup_{n=1}^{\infty} a_n ,$$

and if h is in a_n, then $z_h = f_{c_{n-1,i}}(z_u)$, where u is in a_1 and i is a positive integer such that $h = f_{c_{n-1,i}}(u)$.

OBSERVATION 1. Let M denote a Cantor set on $[0, 1]$. Then there exists a compact point set M' satisfying the following conditions:

(1) M' is a subset of $[0, 1]\times M$.

(2) Each component of M' is an arc or a point.

(3) There are only countably many non-degenerate components of M'.

(4) If U is the collection of all non-degenerate components of M', then $M' - U^*$ is dense in M' and U^* is dense in M'.

(5) If u is in U, then $\overline{U^* - u}$ contains u.

(6) If the components of M' are regarded as points, and the components of M' are denoted by the collection G, then M'/G is a Cantor set.

(7) Let E be the set to which P belongs if and only if P is a point of M which is the left endpoint of the closure of some component of $[0, 1] - -M$. Then there is one and only one non-degenerate component of M' lying on $\{P\} \times [0, 1]$ and U^* is a subset of $E \times [0, 1]$.

Outline of the proof. Let $P_1, P_2, P_3, \ldots$ denote the points of E and let I_{11} denote a subinterval of $[0, 1]$ of length $1/2$. Let $S_1 = \{P_{11}, P_{12}, P_{13}, \ldots\}$ stand for a subset of E such that $P_{11} = P_1$ and P_{11} is the only limit point of S_1. Let $I_{11}, I_{12}, I_{13}, \ldots$ denote a sequence of subintervals of I_{11} such that the length of I_{1n} is $1/2^n$, and

$$\overline{\bigcup_{n=2}^{\infty} (P_{1n} \times I_{1n})} - \bigcup_{n=2}^{\infty} (P_{1n} \times I_{1n}) = P_{11} \times I_{11}.$$

Let P_{21} denote the first element of E not in S_1 and let $S_2 = \{P_{21}, P_{22}, P_{23}, \ldots\}$ denote a subset of $E - S_1$ such that $S_1 \cup P_{21}$ is the set of all limit points of S_2. Let $I_{21}, I_{22}, I_{23}, \ldots$ denote a sequence of intervals such that

for each n, I_{2n} is of length less than $1/3^n$;

if, for each n, P_{1n_1} is the left most point of the set of all points of S_1 to the right of P_{2n}, then I_{2n} is a subset of I_{1n_1};

$$\overline{\bigcup_{n=2}^{\infty} (P_{2n} \times I_{2n})} - \bigcup_{n=2}^{\infty} (P_{2n} \times I_{2n}) = \bigcup_{n=1}^{\infty} (P_{1n} \times I_{in}).$$

Continuing inductively, let

$$M' = \overline{\bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} I_{in}}.$$

THEOREM 2. *If O_1 and O_2 are two 2-cells in the plane having only one point in common, then $O_1 \cup O_2$ is in $\mathscr{K}$.*

Proof. Let M denote a compact continuum in the plane satisfying the conditions of Theorem 1 and let $G, \bar{S}, l, H = h_1, h_2, \ldots$, and $z_1, z_2, z_3, \ldots$ be as described in Theorem 1. Let K denote the collection to which k belongs if and only if, for some positive integer i, k is the closure of a component of $h_i - z_i$. We know that

$$(G-H)^* \cup K^* = \overline{M - \bigcup_{i=1}^{\infty} z_i}$$

and $(G-H) \cup K$ is an upper semi-continuous collection of mutually exclusive arcs filling up the set

$$\overline{M - \bigcup_{i=1}^{\infty} z_i}.$$

Let $z_{n_1}, z_{n_2}, z_{n_3}, \ldots$ denote the arcs of $z_1, z_2, z_3, \ldots$ which do not lie on l and let $z_{m_1}, z_{m_2}, z_{m_3}, \ldots$ denote the elements of $z_1, z_2, z_3, \ldots$ which lie on l.

Let $z'_{n_1}, z'_{n_2}, z'_{n_3}, \ldots$ and $z''_{n_1}, z''_{n_2}, z''_{n_3}, \ldots$ denote two mutually exclusive subsequences of $z_{n_1}, z_{n_2}, z_{n_3}, \ldots$ such that $\bigcup_{i=1}^{\infty} z'_{n_i}$ and $\bigcup_{i=1}^{\infty} z''_{n_i}$ are dense in M.

Let $g_1, g_2, g_3, \ldots$ denote a countable collection of subintervals of $[0, 1]$ such that

$$\lim_{i \to \infty} d(g_i) = 0,$$

$\{\bigcup_{i=1}^{\infty} g_i \times z'_{n_i}\} \times \{0\}$ is dense in

$$M_1 = [\{\bigcup_{i=1}^{\infty} (g_i \times z'_{n_i})\} \cup \{[0, 1] \times [(G-H) \cup K]^*\}] \times \{0\},$$

and $\{\bigcup_{i=1}^{\infty} g_i \times z''_{n_i}\} \times \{1\}$ is dense in

$$M_2 = [\{\bigcup_{i=1}^{\infty} (g_i \times z''_{n_i})\} \cup \{[0, 1] \times [(G-H) \cup K]^*\}] \times \{1\}.$$

Let M' be a set on $\bar{S}' \times [0, 1]$ that satisfies the conditions of Observation 1. Let U denote the collection to which u belongs if and only if one of the following conditions is satisfied:

(1) u is a point of

$$M_1 - [\{M' \cup \bigcup_{i=1}^{\infty} (g_i \times z'_{n_i})\} \times \{0\}]$$

or u is a point of

$$M_2 - [\{M' \cup \bigcup_{i=1}^{\infty} (g_i \times z''_{n_i})\} \times \{1\}];$$

(2) for some positive integer n and some point P of g_n, u is $P\times z_n$ or u is $P\times z_n''$;

(3) for some point P of M', u is the pair $P\times\{0\}$ and $P\times\{1\}$.

U is an upper semi-continuous decomposition of $M_1\cup M_2$, since

$$\lim_{i\to\infty} d(g_i\times z'_{n_i}) = 0, \quad \lim_{i\to\infty} d(g_i\times z''_{n_i}) = 0,$$

and for each positive number ε there are only finitely many components of M' of diameter greater than ε.

$(M_1\cup M_2)\cup U$ is irreducible from

$$\big[\{[0,1]\times A\}\times\{0\}\cup\{[0,1]\times A\}\times\{1\}\big]/U$$

to

$$\big[\{[0,1]\times B\}\times\{0\}\cup\{[0,1]\times B\}\times\{1\}\big]/U,$$

since if P is a point of $(M_1\cup M_2)/U$, and R is a domain containing P, then there exists a positive integer i such that R contains

$$[(g_i\times z'_{n_i})\times\{0\}]/U \qquad \text{or} \qquad [(g_i\times z''_{n_i})\times\{1\}]/U.$$

Let U' be the collection to which u' belongs if and only if, for some element g of $G-H$, u' is

$$\big\{\big[[0,1]\times g\big]\times\{0\}\cup\big[[0,1]\times g\big]\times\{1\}\big\}/U$$

or, for some i, u' is

$$\big[\{[0,1]\times(\overline{h'_{n_i}-z'_{n_i}})\cup(g_i\times z'_{n_i})\}\times\{0\}\cup$$
$$\cup\{[0,1]\times(\overline{h'_{n_i}-z''_{n_i}})\cup(g_i\times z''_{n_i})\}\times\{1\}\big]/U.$$

U' is an upper semi-continuous collection of mutually exclusive continua each homeomorphic to that described in the hypothesis, filling up $(M_1\cup M_2)/U$ so that U' is an arc with respect to its elements.

OBSERVATION 2. If the elements of U that are obtained by taking points of M' and crossing them with $\{0\}$ and $\{1\}$, respectively, are exploded into intervals, then it can be seen that two mutually exclusive 2-cells A and B joined together by an arc intersecting A and B only with its endpoints belongs to $\mathscr{K}$.

OBSERVATION 3. If M is the sum of a finite chain of 2-cells, i.e., the sum of any finite collection of mutually exclusive 2-cells which are joined together by arcs and there is no decomposition U of M such that M/U is a simple closed curve, then M belongs to $\mathscr{K}$.

Reçu par la Rédaction le 14. 10. 1976;
en version modifiée le 14. 3. 1977