COLLOQUIUM MATHEMATICUM

VOL. XXXIII 1975 FASC. 2

PREDICATES AND MEASURES ON BOOLEAN σ-ALGEBRAS

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Let $(A, \vee, \wedge, {}^-, 0, 1)$ be a Boolean σ-algebra (an $\aleph_0$-complete Boolean algebra in the terminology of [2]). The natural partial ordering of A is defined by $x \leqslant y$ if and only if $x \vee y = y$.

For any (countable) subset E of A the notation

$$z = \dot{\bigvee_{x \in E}} x$$

will mean that

$$z = \bigvee_{x \in E} x \quad \text{and} \quad x \wedge y = 0$$

for all distinct $x, y \in E$. Let Π be a *predicate* on A, i.e. a mapping of A into the two-element Boolean algebra $\{0, 1\}$. Π is called *additive* if, for every countable subset E of A such that

$$\bigwedge_{x \in E} \Pi(x) = 1,$$

we have

$$\Pi\big(\dot{\bigvee_{x \in E}} x\big) = 1.$$

Given a predicate Π, we put $(H\Pi)(x) = 1$ if $\Pi(y) = 1$ for all $0 \neq y \leqslant x$, and $(H\Pi)(x) = 0$ otherwise. It is evident that $H\Pi$ is a predicate on A. It follows from the distributive law

$$\big(\bigvee_{x \in E} x\big) \wedge y = \bigvee_{x \in E} (x \wedge y)$$

that if Π is additive, then so is $H\Pi$.

A function $\mu\colon A \to R$ is called a (signed) *measure* if

$$\mu(z) = \sum_{x \in E} \mu(x)$$

for every countable subset E of A such that

$$z = \dot{\bigvee_{x \in E}} x.$$

Here and in the sequel R stands for the set of all real numbers. Given a measure μ, we denote by O_μ the predicate $H\Pi$, where $\Pi(x) = 1$ if $\mu(x) = 0$ and $\Pi(x) = 0$ otherwise. A predicate Π on A satisfying $O_\mu(x) \leqslant \Pi(x)$ for all $x \in A$ is called *μ-continuous*. A predicate Π on A satisfying

$$\Pi(x) \wedge O_\mu(y) \leqslant \Pi(x \vee y) \quad \text{for all } x,\ y \in A$$

is called *μ-stable*.

Given a predicate Π, we put $(H_\mu\Pi)(x) = 1$ if $\Pi(y) = 1$ for all $y \leqslant x$ with $O_\mu(y) = 0$ and $(H_\mu\Pi)(x) = 0$ otherwise.

The purpose of this note is to establish a decomposition theorem for predicates which generalizes (as will be shown later) the well-known decomposition theorems of Hahn, Jordan and Lebesgue.

THEOREM. *If a predicate Π and its negation $\neg\Pi$ are additive, Π is μ-continuous and $\neg\Pi$ is μ-stable, then there exists a $y \in A$ for which $H\Pi(y) = 1$ and $H_\mu(\neg\Pi(\bar{y})) = 1$.*

To prove this Theorem we need some lemmas.

LEMMA 1. *If Π is additive and μ-stable, then $H_\mu\Pi$ is additive.*

Proof. Let

$$y = \dot{\bigvee_{x \in E}} x \quad \text{and} \quad \bigwedge_{x \in E} (H_\mu\Pi)(x) = 1,$$

where E is a countable subset of A. Fix a $z \in A$ with $z \leqslant y$ and $O_\mu(z) = 0$, and put

$$E_1 = \{x \in E\colon\ O_\mu(z \wedge x) = 0\} \quad \text{and} \quad E_2 = E - E_1.$$

Then

$$\bigwedge_{x \in E_1} \Pi(z \wedge x) = 1,$$

so that, by the additivity of Π,

$$\Pi\Big(\dot{\bigvee_{z \in E_1}} (z \wedge x)\Big) = 1.$$

Moreover, the additivity of O_μ yields

$$O_\mu\Big(\dot{\bigvee_{z \in E_2}} (z \wedge x)\Big) = 1.$$

It now follows from the μ-stability of Π that $\Pi(z) = 1$. Therefore, $H_\mu\Pi(y) = 1$ and the proof is complete.

LEMMA 2. *If Π is additive and μ-continuous, then*

$$H_\mu\big(\neg H_\mu(\neg\Pi)\big) \leqslant H\Pi.$$

Proof. Let $x \in A$. We may assume that

$$\big(H_\mu\big(\neg H_\mu(\neg\Pi)\big)\big)(x) = 1.$$

This means that, for every $y \leqslant x$ with $O_\mu(y) = 0$, there exists a $z \leqslant y$ such that $O_\mu(z) = 0$ and $\Pi(z) = 1$. We must then deduce that $\Pi(y) = 1$ for all $y \leqslant x$.

If $O_\mu(y) = 1$, we have $\Pi(y) = 1$ by the μ-continuity of Π. Suppose $O_\mu(y) = 0$. Then one can easily construct, by transfinite induction, a countable set $\{z_a\} \subset A$ such that $z_a \leqslant y$, $O_\mu(z_a) = 0$, $\Pi(z_a) = 1$, and $O_\mu(y \wedge \bar{z}) = 1$, where $z = \dot{\bigvee_a} z_a$. Since Π is μ-continuous, $\Pi(y \wedge \bar{z}) = 1$, so that, by the additivity of μ, $\Pi(y) = 1$.

Proof of the Theorem. If there exists no $x \in A$ with $(H_\mu \neg \Pi)(x) = 1$ and $O_\mu(x) = 0$, then, by Lemma 2, the assertion holds for $y = 1$.

Suppose there exists an $x \in A$ with $\big(H_\mu(\neg \Pi)\big)(x) = 1$ and $O_\mu(x) = 0$. Then one can easily construct, by transfinite induction, a countable set $\{z_a\} \subset A$ such that

$$\big(H_\mu(\neg \Pi)\big)(z_a) = 1, \quad O_\mu(z_a) = 0 \quad \text{and} \quad \big(H_\mu(\neg \Pi)\big)(u) = 0$$

whenever $u \wedge z = 0$ and $O_\mu(u) = 0$, where $z = \dot{\bigvee_a} z_a$. Put $y = \bar{z}$. Clearly, we then have

$$\big(H_\mu(\neg H_\mu(\neg \Pi))\big)(y) = 1,$$

so that, according to Lemma 2, $(H\Pi)(y) = 1$. Moreover, by Lemma 1, $\big(H_\mu(\neg \Pi)\big)(\bar{y}) = 1$. Thus the Theorem is proved.

Remark. It is easy to verify that an element y satisfying the conditions of the Theorem is defined μ-almost uniquely. More precisely, if there exists another element $y' \in A$ for which we also have $(H\Pi)(y') = 1$ and $\big(H_\mu(\neg \Pi)\big)(y') = 1$, then

$$O_\mu\big((y \wedge \bar{y}') \vee (\bar{y} \wedge y')\big) = 1.$$

Let us give some important applications of the Theorem.

First we consider the following predicate: $\Pi(x) = 1$ iff $\mu(x) \geqslant 0$, where μ is a measure on A. Clearly, all assumptions of the Theorem are satisfied. Hence there exists a $y \in A$ such that $\mu(z) \geqslant 0$ if $z \leqslant y$, and $\mu(z) \leqslant 0$ if $z \leqslant \bar{y}$. Thus the elements y and $\bar{y}$ form a Hahn decomposition of 1 with respect to μ (see [1], p. 121). In view of the Remark, such a decomposition is μ-almost unique. Put $\mu_+(x) = \mu(x \wedge y)$ and $\mu_-(x) = -\mu(x \wedge \bar{y})$ for $x \in A$. Clearly, μ_+ and μ_- are non-negative (finite) mutually singular measures on A. Moreover, we have

$$\mu(x) = \mu_+(x) - \mu_-(x), \tag{1}$$

which is the Jordan decomposition of μ (see [1], p. 123).

Now we can define the variation of the measure μ on an $x \in A$ as the sum

$$|\mu|(x) = \mu_+(x) + \mu_-(x).$$

The function $|\mu|$ is a positive (finite) measure on A and $|\mu|(x) = 0$ is equivalent to $O_\mu(x) = 1$.

Next we consider a predicate Π defined by $\Pi(x) = 1$ iff $O_\mu(x) \leqslant O_\nu(x)$, where μ and ν are measures on A. We write

$$\pi(x) = |\mu|(x) + |\nu|(x).$$

The predicate Π is additive and π-stable. An application of the Theorem gives immediately the decomposition

$$\nu(x) = \nu_a(x) + \nu_s(x), \tag{2}$$

where $\nu_a(x) = \nu(x \wedge y)$ and $\nu_s(x) = \nu(x \wedge \bar{y})$. It is easy to verify that ν_a is absolutely continuous and ν_s is singular with respect to μ. Thus (2) is the Lebesgue decomposition of ν relative to μ (see [1], p. 134).

The author is thankful to I. M. Gelfand and H. Rasiowa for a discussion of these problems.

REFERENCES

[1] P. Halmos, *Measure theory*, New York 1950.
[2] R. Sikorski, *Boolean algebras*, Berlin - New York 1964.

Reçu par la Rédaction le 8. 9. 1972;
en version modifiée le 12. 1. 1973 et le 27. 7. 1973