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# OSCILLATORY INTEGRALS ASSOCIATED TO FOLDING CANONICAL RELATIONS

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**1. Introduction.** We shall consider oscillatory integral operators

$$T_\lambda f(x) = \int_{\mathbf{R}^n} e^{i\lambda\phi(x,y)}\beta(x,y)f(y)dy, \quad \lambda > 0, \tag{1.1}$$

where $\beta \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ and where the phase function $\phi \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ is real. If

$$\det\left(\frac{\partial^2 \phi}{\partial x_j \partial y_k}\right) \neq 0, \tag{1.2}$$

it was proved in Hörmander [4] that

$$\|T_\lambda f\|_{L^{p'}(\mathbf{R}^n)} \leq C\lambda^{-n/p'}\|f\|_{L^p(\mathbf{R}^n)}, \quad 1 \leq p \leq 2, \quad 1/p + 1/p' = 1. \tag{1.3}$$

By taking $\phi = \langle x, y\rangle$, one sees that this inequality implies the Hausdorff–Young inequality. It is also not hard to argue that the decay in $\lambda$ is optimal.

The purpose of this paper is to study oscillatory integral operators when the hypothesis (1.2) is relaxed. Since $L^p \to L^{p'}$ estimates follow from interpolating between $L^2 \to L^2$ and $L^1 \to L^\infty$ ones (the latter are trivial), the real issue is to obtain the optimal decay of $\|T_\lambda\|_{L^2 \to L^2}$ in $\lambda$.

Before stating our hypotheses, it is instructive to state (1.2) in an equivalent form. Given any phase function, one can define the associated canonical relation

$$C_\phi = \{\, (x, \phi'_x(x,y), y, -\phi'_y(x,y))\,\}. \tag{1.4}$$

Since we are assuming that $\phi$ is real and $C^\infty$, it follows that $C_\phi$ is a smooth Lagrangean submanifold of $T^*\mathbf{R}^n \times T^*\mathbf{R}^n$, when the latter is endowed with the usual symplectic form $d\xi \wedge dx - d\eta \wedge dy$. Here $T^*\mathbf{R}^n$ of course denotes the cotangent bundle of $\mathbf{R}^n$. The convention of including the minus sign in the last factor of (1.4) is used in the theory of Fourier integral operators

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and it will also be useful for us since it simplifies the composition formulas to follow.

Using the canonical relation $C_\phi$, we can finally reformulate (1.2). In fact, if

$$\Pi_1(x,\xi,y,\eta) = (x,\xi), \qquad \Pi_2(x,\xi,y,\eta) = (y,\eta)$$

are the projection operators onto the two factors, then (1.2) is equivalent to the condition that $\Pi_j : C_\phi \to T^*\mathbf{R}^n$, $j = 1,2$, are local diffeomorphisms. In this nondegenerate case the condition is symmetric: if one of the projection operators from $C_\phi$ to $T^*\mathbf{R}^n$ is a local diffeomorphism, then so is the other one.

In many cases, such as in the study of restriction theorems or Bochner–Riesz theorems (see [1], [9]), one wishes to prove optimal estimates for the oscillatory integral operators when the mappings $\Pi_j : C_\phi \to T^*\mathbf{R}^n$ are allowed to be singular. We shall be concerned with the case where the mappings may have the simplest type of singularities. Recall that a $C^\infty$ map $f$ between $C^\infty$ manifolds $X$ and $Y$ is said to have a *Whitney fold* at $x_0 \in X$ if (i) dim Ker $f'(x_0) =$ dim Coker $f'(x_0) = 1$, and (ii) the Hessian of $f$ at $x_0$ is nonzero. In this context the *Hessian* is the quadratic form

$$\operatorname{Ker} f'(x_0) \ni \eta \to \langle f''(x_0)\eta,\eta\rangle \in \operatorname{Coker} f'(x_0).$$

The standard example of a map with a folding singularity is the map $f(x) = (x_1,\ldots,x_{n-1},x_n^2)$. In fact, it is well known that if $f$ has a folding singularity at $x_0$, then local coordinates can be chosen around $x_0$ and $f(x_0)$ so that $f$ takes this form. (See [5, Vol. III, pp. 492–493].)

We can now state our main result.

THEOREM 1.1. *Suppose that each of the projections $\Pi_j : C_\phi \to T^*\mathbf{R}^n$, $j = 1,2$, has at most folding singularities. Then*

$$\|T_\lambda f\|_{L^2(\mathbf{R}^n)} \le C\lambda^{-n/2+1/6}\,\|f\|_{L^2(\mathbf{R}^n)}, \tag{1.5}$$

*and this result is always sharp if $\beta(x_0,y_0) \ne 0$ below some folding point $(x_0,\phi'_x,y_0,-\phi'_y)$.*

There is a homogeneous version of (1.5) which is due to Melrose [6]: If $F \in I^\mu(X,Y;C')$ is a Fourier integral operator of order $\mu$, and if each of the projections from its canonical relation to $T^*X\backslash 0$ and $T^*Y\backslash 0$ has at most folding singularities, then $F : L^2_{k+\mu,\mathrm{comp}}(Y) \to L^2_{k-1/6,\mathrm{loc}}(X)$. This result is sharp, and like (1.5) it reflects a loss of 1/6 derivatives compared to the nondegenerate case.

The proof of the restriction theorem of Zygmund [9] also led us to study folding oscillatory integral operators. Recall that at a certain stage in the proof of the restriction theorem for $\mathbf{R}^2$ one is led to estimating oscillatory integrals of the form

$$\int_0^{2\pi}\int_0^{2\pi} e^{i\lambda\langle x,(\cos t_1+\cos t_2,\sin t_1+\sin t_2)\rangle} f(t_1)f(t_2)\,dt. \tag{1.7}$$

The phase function $\phi(x,t)$ here has the property that the projection from $C_\phi$ to the second factor of $T^*\mathbf{R}^2 \times T^*\mathbf{R}^2$ has folding singularities, while the projection onto the first factor has a different type of singularity when $t_1 = t_2$. Thus operators of this form do not quite fall into the scope of Theorem 1.1, which is well since it is known that the $L^2$ norm of (1.7) (over, say, the unit ball) is only $\leq C\lambda^{-1+1/4}(\int\int |f(t_1)f(t_2)|^2\,dt)^{1/2}$ in general. In the proof of the restriction theorem, on the other hand, one does not try to estimate the $L^2$ norm of (1.7) directly in terms of the $L^2$ norm of $f$. Instead, the argument of [9] uses the fact that the oscillatory integral (1.7) has a very special form since its integrand is symmetric in $t_1$ and $t_2$.

Returning to Theorem 1.1, the model case occurs when the phase function of $T_\lambda$ is

$$\phi = \phi_0 = \langle x', y'\rangle + (x_n - y_n)^3, \qquad x' = (x_1, \ldots, x_{n-1}). \tag{1.8}$$

Here the folding points occur when $x_n = y_n$. In the proof of our result, we shall first show that operators with this phase function satisfy the desired estimate (1.5). The next step will be a reduction which shows that the estimate for general operators follows from this special case. The main ingredient in this part of the argument is a result of Melrose and Taylor [7] which says that, near a folding point, one can make a symplectic change of variables transforming a general canonical relation (1.4) to the model canonical relation $C_{\phi_0}$. One of course makes use of this change of variables lemma by conjugating $T_\lambda$ by certain nondegenerate oscillatory integral operators. This argument is similar to ones involving Egorov's theorem since the nondegenerate oscillatory integral operators one uses are essentially dyadic Fourier integral operators.

**2. Model case.** The purpose of this section is to prove a special case of Theorem 1.1. In the next section we shall see that the general case follows from this special case and the estimate (1.3) for nondegenerate oscillatory integral operators.

PROPOSITION 2.1. *Let $\beta \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ and set*

$$T_\lambda^0 f(x) = \int_{\mathbf{R}^n} e^{i\lambda\phi_0(x,y)}\beta(x,y)f(y)\,dy, \tag{2.1}$$

*where $\phi_0$ is the phase function defined in* (1.8). *Then*

$$\|T_\lambda^0 f\|_{L^2(\mathbf{R}^n)} \leq C\lambda^{-n/2+1/6}\|f\|_{L^2(\mathbf{R}^n)}. \tag{2.2}$$

The proof is very simple. Let us assume for a moment the following:

LEMMA 2.2. *Suppose that $T_\lambda^0$ is defined as above. If in addition $\beta(x,y) \equiv 0$ when $x_n = y_n$, then, for large $\lambda$,*

$$\|T_\lambda^0 f\|_{L^2(\mathbf{R}^n)} \le \lambda^{-n/2}(\log\lambda)^{1/2}\,\|f\|_{L^2(\mathbf{R}^n)}. \tag{2.3}$$

Since the decay in (2.3) is much faster than in (2.2), we conclude that, in the proof of (2.2), we may replace $\beta$ by a cutoff function of the form $\beta_1(x',y')\cdot\beta_2(x_n-y_n)$, where $\beta_1 \in C_0^\infty(\mathbf{R}^{n-1}\times\mathbf{R}^{n-1})$ and $\beta_2 \in C_0^\infty(\mathbf{R})$. Then we have

$$T_\lambda^0 f(x',x_n) = \int_{\mathbf{R}} e^{i\lambda(x_n-y_n)^3}\beta_2(x_n-y_n)S_\lambda(f(\,\cdot\,,y_n))(x')dy_n,$$

with

$$S_\lambda(f(\,\cdot\,,y_n))(x') = \int_{\mathbf{R}^{n-1}} e^{i\lambda\langle x',y'\rangle}\beta_1(x',y')f(y',y_n)dy'.$$

The result for the nondegenerate case, (1.3), yields

$$\|S_\lambda(f(\,\cdot\,,y_n))\|_{L^2(\mathbf{R}^{n-1})} \le C\lambda^{-(n-1)/2}\|f(\,\cdot\,,y_n)\|_{L^2(\mathbf{R}^{n-1})}. \tag{2.4}$$

By taking Fourier transforms in the $x_n$-variable, we have

$$\int_{\mathbf{R}} T_\lambda^0 f(x',x_n)e^{i\lambda x_n\xi_n}\,dx_n = m_\lambda(\xi_n)\cdot\int_{\mathbf{R}} S_\lambda(f(\,\cdot\,,y_n))(x')e^{iy_n\xi_n}\,d\xi_n,$$

where $m_\lambda(\xi_n) = \int_{\mathbf{R}} e^{i[\xi_n t+\lambda t^3]}\beta_2(t)\,dt$. It is well known that $|m_\lambda(\xi_n)| \le C\lambda^{-1/3}$. By using Plancherel's theorem, this immediately leads us to

$$\int_{\mathbf{R}} |T_\lambda^0 f(x',x_n)|^2\,dx_n \le C\lambda^{-2/3}\int_{\mathbf{R}} |S_\lambda(f(\,\cdot\,,y_n))(x')|^2\,dx_n.$$

If we now integrate in the $x'$-variable and apply (2.4), we get

$$\int_{\mathbf{R}^n} |T_\lambda^0 f(x',x_n)|^2\,dx \le C\lambda^{-(n-1)-2/3}\int_{\mathbf{R}^n} |f(x)|^2\,dx.$$

Proposition 2.1 is proved. ∎

Proof of Lemma 2.2. If we set $\tilde\beta(x,y) = \beta(x,y)$ when $x_n \ge y_n$, and 0 otherwise, then the desired estimate would follow if we could show that the operators

$$\tilde T_\lambda^0 f(x) = \int_{\mathbf{R}^n} e^{i\lambda\phi_0(x,y)}\tilde\beta(x,y)f(y)\,dy$$

also satisfy an analogue of (2.3).

To prove this, we shall use the argument of [4]. After squaring the $L^2$ norm and taking adjoints, it is not difficult to see that this is equivalent to the estimate

$$\|(\widetilde{T}_\lambda^0)^* \widetilde{T}_\lambda^0 f\|_{L^2} \le C\lambda^{-n}\log\lambda\|f\|_{L^2}. \tag{2.3'}$$

The kernel of the operator here is

$$K_\lambda(x,y) = \int_{\mathbf{R}^n} e^{i\lambda[\langle x'-y',z'\rangle+(x_n-z_n)^3-(y_n-z_n)^3]}\overline{\widetilde{\beta}(x,z)}\,\widetilde{\beta}(y,z)\,dz.$$

Observe that on the support of the integrand

$$|(\partial/\partial z_n)[(x_n-z_n)^3-(y_n-z_n)^3]| \approx |x_n-y_n|(|x_n-z_n|+|y_n-z_n|).$$

Thus, since we are assuming that $\widetilde{\beta}=0$ when $x_n=z_n$, an easy integration by parts argument shows that

$$|K_\lambda(x,y)| \le C_N(1+\lambda|x'-y'|)^{-N}(1+\lambda|x_n-y_n|)^{-1}$$

for any $N$. Since this estimate implies that the $L^1$ norm of the kernel with respect to either of the variables is $O(\lambda^{-n}\log\lambda)$, we get (2.3′).

**3. Reduction to model case.** To prove Theorem 1.1 we shall use the special case, Proposition 2.1, plus the following change of coordinates lemma of Melrose and Taylor [7].

LEMMA 3.1. *Let $C\subset T^*\mathbf{R}^n\times T^*\mathbf{R}^n$ be a canonical relation. Assume that at $C\ni(s_1,s_2)$ both of the projection operators $\Pi_j$, $j=1,2$, have folding singularities. Then there are local symplectic coordinates $(x,\xi)$ near $s_1$ and $(y,\eta)$ near $s_2$ so that $C$ becomes*

$$C_{\phi_0} = \{(x,y',3(x_n-y_n)^2;\ y,-x',3(x_n-y_n)^2)\}. \tag{3.1}$$

*In other words, in the new symplectic coordinates, $C$ is the canonical relation associated to the model phase function $\phi_0=\langle x',y'\rangle+(x_n-y_n)^3$.*

Remark. Lemma 3 says that there are canonical transformations $\chi_1$, $\chi_2$ so that (locally)

$$C_{\phi_0} = \{(x,\xi,y,\eta) : (x,\xi)=\chi_1(w,\gamma),(y,\eta)=\chi_2(z,\zeta),\ (w,\gamma,z,\zeta)\in C\}. \tag{3.1'}$$

Unlike the usual situation (i.e. when one uses Egorov's theorem), the canonical transformations need not be homogeneous.

To apply this lemma, we shall need to understand the composition of $T_\lambda$ with certain operators that are nonhomogeneous versions of nondegenerate Fourier integral operators. To this end, suppose that $\chi$ is a canonical transformation. Then

$$G=\{(x,\xi,y,\eta):\ (x,\xi)=\chi(y,\eta)\} \tag{3.2}$$

is a Lagrangean submanifold of $T^*\mathbf{R}^n \times T^*\mathbf{R}^n$. If $G \subset (T^*\mathbf{R}^n\backslash 0)\times(T^*\mathbf{R}^n\backslash 0)$, then it is well known that we can choose local coordinates around a given $x_0$ so that $G \ni (x,\xi,y,\eta) \to (x,\eta)$ is a diffeomorphism. (See [3, pp. 153–154].) In this situation $G$ is described by a generating function $S(x,\eta)$, i.e.

$$G = \{(x, \nabla_x S(x,\eta), \nabla_\eta S(x,\eta), \eta)\}. \tag{3.2'}$$

In other words, $G$ is parameterized by the phase function

$$\Psi = S(x,\xi) - \langle y, \xi\rangle. \tag{3.3}$$

Let $F_\lambda$ be the following operator associated to (3.2)–(3.3):

$$F_\lambda g(x) = \lambda^n \int\int e^{i\lambda\Psi(x,y,\xi)}\alpha(x,y,\xi)g(y)\,d\xi dy, \tag{3.4}$$

where $\alpha \in C_0^\infty(\mathbf{R}^{3n})$.

The next result describes the composition of the operators in (1.1) and (3.4).

LEMMA 3.2. *$T_\lambda \circ F_\lambda$ is an oscillatory integral operator of the form* (1.1) *whose phase function $\tilde{\phi}$ has canonical relation*

$$C_{\tilde{\phi}} = C_\phi \circ G = \{(x,\xi,y,\eta) : \textit{for some } (z,\zeta), (x,\xi,z,\zeta) \in C_\phi, (z,\zeta,y,\eta) \in G\}.$$

*Similarly, $F_\lambda \circ T_\lambda$ is an oscillatory integral operator of the same form whose phase function has the canonical relation $G \circ C_\phi$. The amplitude, $\beta$, depends on $\lambda$, but it belongs to a bounded subset of $C_0^\infty$ as $\lambda \to \infty$.*

Proof. Since results of this type are essentially in [3], we shall just sketch the proof. The kernel of the composition is

$$\lambda^n \int\int e^{i[\lambda\phi(x,z)+S(y,\xi)-\langle z,\xi\rangle]}\overline{\alpha(y,z,\xi)}\,\beta(x,z)\,d\xi\,dz.$$

Stationary points occur when $\nabla_\xi\Phi = 0$ and $\nabla_z\Phi = 0$, if $\Phi$ is the phase function in the formula for the composition. The stationary points are nondegenerate since

$$\det\Big(\frac{\partial^2 S}{\partial z_j \partial \xi_k}\Big) \neq 0.$$

Consequently, the result follows from van der Corput's lemma.

LEMMA 3.3. *Suppose that $\alpha(x_0,y_0,\xi_0) \neq 0$. Then $F_\lambda F_\lambda^*$ has kernel*

$$\lambda^n \int e^{i\lambda\langle x-y,\xi\rangle}\tilde{\alpha}(x,y,\xi)\,d\xi,$$

*where $\tilde{\alpha}(x_0,y_0,\xi_0) \neq 0$ and $\tilde{\alpha} \in C_0^\infty(\mathbf{R}^{3n})$.*

Proof. Similar to proof of Lemma 3.2.

If we use the fact that the norm of $F_\lambda^*$ is the square root of the norm of $F_\lambda F_\lambda^*$, then Lemma 3.3 yields:

COROLLARY 3.4. *If $S(y,\xi)$ and $\alpha$ as above are fixed, then one has the uniform bounds $\|F_\lambda^* f\|_2 \le C\|f\|_2$.*

We have finally built up enough machinery to prove Theorem 1.1. After perhaps multiplying on the left and right by $e^{i\lambda\langle x,\xi\rangle}$, for appropriate fixed $\xi \in \mathbf{R}^n$, we may always assume that $C_\phi \subset (T^*\mathbf{R}^n\backslash 0) \times (T^*\mathbf{R}^n\backslash 0)$. Let then $\chi_1$ and $\chi_2$ be the canonical transformations given by Lemma 3.1. After choosing the appropriate local coordinates, we can find $F^1_\lambda$ and $F^2_\lambda$ as above so that $F^2_\lambda T_\lambda F^1_\lambda$ is a model oscillatory integral with phase function $\phi_0$ as in (1.8). After perhaps contracting the support of the symbol of $T_\lambda$, we can arrange things so that $(F^2_\lambda)^* F^2_\lambda T_\lambda F^1_\lambda (F^1_\lambda)^* = T_\lambda + R_\lambda$, where $R_\lambda$ has $L^2$ operator norm $O(\lambda^{-N})$ for any $N$.

Thus, we conclude that Theorem 1.1 follows from Proposition 2.1 and Corollary 3.4.

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