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ON σ-ORTHODISTRIBUTIVITY

BY

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In this note it is shown that any σ-orthomodular poset $\mathscr{L}$ with the property of σ-orthodistributivity can weakly be embedded into a Boolean σ-algebra (Theorem 4). Moreover, if $\mathscr{L}$ is a completely σ-orthomodular poset with the property of σ-orthodistributivity, then $\mathscr{L}$ is embeddable into a Boolean σ-algebra (Theorem 5).

Let L be a non-empty set, partially ordered by the relation $\leqslant$. Let $a \mapsto a'$ map L into L. One says that $\mathscr{L} = \langle L, \leqslant, ' \rangle$ is a *σ-orthomodular partially ordered set* (*σ-orthomodular poset*) if the following conditions are fulfilled:

(1) $(a')' = a$;

(2) if $a \leqslant b$, then $b' \leqslant a'$;

(3) if $a_1, a_2, \ldots, a_n, \ldots$ belong to L $(n \in N)$ and $a_i \leqslant a_j'$ for any i, j $(i \neq j)$, then the least upper bound (l.u.b.) a of the set $\{a_n : n \in N\}$ exists in L, i.e., $a = a_1 \cup a_2 \cup \ldots$;

(4) $a \cup a' = b \cup b'$ for any $a, b \in L$;

(5) if $a \leqslant b$, then $b = a \cup (b' \cup a)'$.

In our further considerations we shall use interchangeably the terms "σ-orthomodular poset" and "quantum logic".

We have the following consequences of conditions (1)-(5).

If $a \leqslant b'$, then a is orthogonal to b and we write $a \perp b$.

It follows from (1) and (2) that $a \leqslant b'$ is equivalent to $b \leqslant a'$, i.e. the orthogonality relation $\perp$ is symmetric. Notice that $a \perp a'$.

Condition (3) says that if $a_1, a_2, \ldots$ are pairwise orthogonal, then the l.u.b. $a = a_1 \cup a_2 \cup \ldots$ exists in L. Then we write $a = a_1 + a_1 + \ldots$ and this means that $a_i \perp a_j$ for $i \neq j$ and $a = a_1 \cup a_2 \cup \ldots$

Condition (4) says that the greatest element 1 (the unit element) exists in L and $1 = a \cup a'$ for every $a \in L$. Hence, by (2), the least element 0 (the zero element) exists in L, and $0 = 1'$.

Taking $a_{k+1} = a_{k+2} = \ldots = 0$ in (3), we see that finite sums of orthogonal elements $a = a_1 + a_2 + \ldots + a_k$ exist in L.

If $a \leqslant b$, then $a \perp b'$; hence $b' \cup a$ exists.

It follows then from (1) and (2) that $(b' \cup a)' = b \cap a'$. But $a \perp b \cap a'$; hence, by (3), $a \cup (b' \cup a)'$ exists.

Condition (5) claims equality and no existence. It is a weakened form of the law of modularity and is referred to as the *law of orthomodularity* or the *law of weak modularity.*

Every set or sequence consisting of pairwise orthogonal elements of L is called an *orthogonal set* or *orthogonal sequence*, respectively.

We admit the following notation: algebraic objects, e.g., Boolean algebras, quantum logics, are denoted by script capital letters $\mathscr{A}, \mathscr{B}, \ldots, \mathscr{L}$ with subscripts, if needed, while the carriers of these objects are denoted by the same italic capital letters $A, B, \ldots, L$.

Every Boolean algebra is an example of quantum logic. Generally, $\mathscr{L}$ need not be a Boolean algebra or even a lattice. However, it may happen that a certain subset $A \subseteq L$ has the properties: if $a, b \in A$, then $a \cup b$ and $a \cap b$ exist in L and belong to A, and if $a \in A$, then $a' \in A$. A system $\mathscr{A} = \langle A; \cup, \cap, ' \rangle$ forms then a lattice with orthocomplementation. Moreover, if $\mathscr{A}$ is distributive, then $\mathscr{A} = \langle A; \cup, \cap, ' \rangle$ is a Boolean algebra contained in $\mathscr{L}$. Observe that then for any two elements $a, b \in A$ the following condition holds in $\mathscr{L}$: $a \perp b$ iff $a \cap b = 0$. It follows from the above that if $\mathscr{A}$ is a Boolean subalgebra in a quantum logic $\mathscr{L}$ and for any sequence $a_1, a_2, \ldots, a_n, \ldots$ of mutually disjoint elements in A (hence orthogonal in L) the l.u.b. $a = a_1 + a_2 + \ldots$ (which exists in L) belongs to A, then $\mathscr{A}$ is a Boolean σ-algebra. Then we say that $\mathscr{A}$ is a Boolean σ-subalgebra in $\mathscr{L}$. Thus we arrive at the following definition:

Definition 1 (see, e.g., [5]). Let A be a subset of a quantum logic $\mathscr{L}$. Then $\mathscr{A} = \langle A; \cup, \cap, ' \rangle$ is a *Boolean σ-subalgebra* in $\mathscr{L}$ with respect to the operations $\cup, \cap, '$ if

(a) for any $a, b \in A$ the l.u.b. $a \cup b$ and the g.l.b. (greatest lower bound) $a \cap b$ exist in L and belong to A;

(b) if $a \in A$, then $a' \in A$;

(c) $\langle A, \cup, \cap \rangle$ is a distributive lattice;

(d) if $a_i \in A$ for $i = 1, 2, \ldots$ and $a_i \perp a_j$ for $i \neq j$, then $a_1 + a_2 + \ldots \in A$.

We admit the following definition:

Definition 2 (see [5]). Let $\mathscr{L}$ be a quantum logic. A subset $X \subseteq L$ is *compatible* if there exists a Boolean σ-subalgebra $\mathscr{A}$ in $\mathscr{L}$ such that $X \subseteq A$.

If $X = \{a, b\}$ and X is compatible, then we write $a \leftrightarrow b$.

THEOREM 1 [5]. *Let $\mathscr{L}$ be a quantum logic and $a, b \in L$. Then $a \leftrightarrow b$ iff there exist pairwise orthogonal elements $a_0, b_0, c_0 \in L$ such that $a = a_0 + c_0$ and $b = b_0 + c_0$.*

Remark. In [1] we have defined *quantum logics* as σ-orthomodular posets satisfying additionally the following condition:

For any $a, b, c \in L$ if a, b, c are pairwise compatible, then $a \cup b \leftrightarrow c$.

Let $\mathscr{L}$ be a quantum logic. Let $\{a_n\}_{n \in N}$ be an orthogonal infinite sequence of elements of L ($a_n \neq 0$, $n = 1, 2, \ldots$) such that $a_1 + a_2 + \ldots = 1$. Notice that $a_1, a_2, \ldots$ are pairwise different. For any set $N_0 \subseteq N$ let

$$a_{N_0} \stackrel{\mathrm{df}}{=} \sup\{a_n : n \in N_0\}.$$

Let $A = \{a_{N_0} : N_0 \in 2^N\}$. If $N_0 = \emptyset$, we admit $a_\emptyset = 0$. The following equalities hold true:

(*) $a'_{N_0} = a_{N-N_0}$,

(**) $a_{N_1 \cup N_2} = a_{N_1} \cup a_{N_2}$,

(***) $a_{N_1 \cap N_2} = a_{N_1} \cap a_{N_2}$.

(*) For any $n \in N - N_0$, $a_n \perp a_{N_0}$, i.e., $a_n \leqslant a'_{N_0}$. Hence $a_{N-N_0} \leqslant a'_{N_0}$. From (5) we obtain

$$a'_{N_0} = a_{N-N_0} \cup (a_{N_0} \cup a_{N-N_0})'.$$

But $a_{N_0} \cup a_{N-N_0} = 1$. Hence $a'_{N_0} = a_{N-N_0} \cup 0 = a_{N-N_0}$.

(**) Obviously, $a_{N_i} \leqslant a_{N_1 \cup N_2}$ $(i = 1, 2)$. Let $b \in L$ and $a_{N_1} \leqslant b$, $a_{N_2} \leqslant b$. Then $a_n \leqslant b$ for any $n \in N_1 \cup N_2$. Hence $a_{N_1 \cup N_2} \leqslant b$.

(***) follows from (*) and (**).

Moreover, $N_0 \mapsto a_{N_0}$ $(N_0 \in 2^N)$ is a one-to-one mapping. Thus we have proved that $\mathscr{A} = \langle A; \cup, \cap, ' \rangle$ is a Boolean algebra in $\mathscr{L}$ isomorphic to the field of all subsets of N (N — natural numbers). Notice that

$$a_{N_1} \perp a_{N_2} \text{ in } L \quad \text{iff} \quad N_1 \cap N_2 = \emptyset.$$

Hence $\mathscr{A}$ is a Boolean σ-subalgebra in $\mathscr{L}$.

Let $b_1, b_2, \ldots$ be an infinite orthogonal sequence ($b_n \neq 0$, $n = 1, 2, \ldots$). If $b_1 + b_2 + \ldots \neq 1$, then we put $b_0 = (b_1 + b_2 + \ldots)'$. Thus the sequence $b_0, b_1, b_2, \ldots$ has the property: $b_0 + b_1 + b_2 + \ldots = 1$.

We use a reasoning similar to that in the case where a sequence $a_1, a_2, \ldots, a_k$ is finite.

Hence we obtain

THEOREM 2. *Let $\mathscr{L}$ be a quantum logic. Then every countable orthogonal set in L is compatible.*

Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be quantum logics. The mapping $h: L_1 \mapsto L_2$ is said to be a *homomorphism* provided that

(i) $a \leqslant b$ implies $ha \leqslant hb$;

(ii) $h(a') = (ha)'$;

(iii) for any orthogonal and countable sequence $a_1, a_2, \ldots$ in L_1

$$h(a_1 + a_2 + \ldots) = ha_1 + ha_2 + \ldots$$

If $h: L_1 \mapsto L_2$ is a homomorphism and $a \leftrightarrow b$ $(a, b \in L_1)$, then

$$ha \leftrightarrow hb \quad \text{and} \quad h(a \cup b) = ha \cup hb.$$

Moreover, if 1 is the unit in L_1, then $h1$ is the unit in L_2.

A homomorphism $h: L_1 \mapsto L_2$ is said to be a *weak embedding* provided that, for any $a, b \in L_1$, if $a \neq b$ and $a \leftrightarrow b$ in L_1, then $ha \neq hb$. A one-to-one homomorphism is an embedding. An embedding h of type "onto" and preserving order, i.e., such that $a \leqslant b$ iff $ha \leqslant hb$ $(a, b \in L_1)$, is said to be an *isomorphism.*

We assume that the reader is familiar with the theory of Boolean σ-products. Detailed discussion of these problems can be found in [6]. Here, for the sake of clarity of our further considerations, let us only recall some notions.

Let $\{\mathscr{A}_\lambda\}_{\lambda \in \Lambda}$ be a family of Boolean σ-algebras. The pair

$$\{\{i_\lambda\}_{\lambda \in \Lambda}, \mathscr{B}\} \tag{6}$$

is said to be a *Boolean σ-product* of the algebras $\{\mathscr{A}_\lambda\}_{\lambda \in \Lambda}$ provided that

(a) $\mathscr{B}$ is a Boolean σ-algebra;

(b) for any $\lambda \in \Lambda$, i_λ is an embedding of the algebra $\mathscr{A}_\lambda$ into $\mathscr{B}$;

(c) the indexed family $\{i_\lambda(\mathscr{A}_\lambda)\}_{\lambda \in \Lambda}$ of subalgebras of $\mathscr{B}$ is σ-independent in $\mathscr{B}$;

(d) the algebra $\mathscr{B}$ is σ-generated by the set-theoretical sum of universes of all algebras $i_\lambda(\mathscr{A}_\lambda)$ $(\lambda \in \Lambda)$.

Let (6) be a σ-product of σ-algebras $\{\mathscr{A}_\lambda\}_{\lambda \in \Lambda}$. Let

$$B_\lambda = i_\lambda(A_\lambda) \quad \text{and} \quad B_0 = \bigcup_{\lambda \in \Lambda} B_\lambda$$

(B_λ — the universe of $\mathscr{B}_\lambda$). B_0 is a partially ordered set with respect to the order generated by the order $\leqslant$ of the algebra $\mathscr{B}$. The set B_0 is also closed under the operation of Boolean complementation. By σ-independence of the algebras $\mathscr{B}_\lambda$, the system $\mathscr{B}_0 = \langle B_0, \leqslant, ' \rangle$ forms a quantum logic. Moreover, if $x_1, x_2, \ldots$ is any orthogonal sequence in B_0, then the l.u.b. $x_1 + x_2 + \ldots$ in the quantum logic $\mathscr{B}_0$ is equal to its l.u.b. in the algebra $\mathscr{B}$.

Let $\mathscr{L}$ be a quantum logic and let $\{\mathscr{A}_\lambda\}_{\lambda \in \Lambda}$ be the family of all Boolean σ-algebras in $\mathscr{L}$. Observe that

$$\bigcup_{\lambda \in \Lambda} A_\lambda = L$$

(A_λ — the carrier of $\mathscr{A}_\lambda$). Let (6) be a Boolean σ-product of the algebras $\{\mathscr{A}_\lambda\}_{\lambda \in \Lambda}$. Let $\mathscr{B}_0 = \langle B_0; \leqslant, ' \rangle$ be the quantum logic defined as above. We define a relation $\sim$ in B_0 as follows:

$x \sim y$ iff there exist $\lambda_1, \lambda_2 \in \Lambda$ and an element $a \in L$ such that $x = i_{\lambda_1} a$ and $y = i_{\lambda_2} a$.

For each $x \in B_0$ there exists exactly one $a \in L$ such that $x = i_\lambda a$. Moreover, if $x \neq 0$ and $x \neq 1$ in B_0, then there exists exactly one λ such that $x = i_\lambda a$. These remarks result from the facts that i_λ is an isomorphism for every $\lambda \in \Lambda$ and that the algebras $\{\mathscr{B}_\lambda\}_{\lambda \in \Lambda}$ are σ-independent.

$\sim$ is an equivalence relation in B_0. Each layer $|x|$ in $B_0/\sim$ is uniquely determined by a certain element $a \in L$, namely by that element a for which $x = i_\lambda a$. Moreover, $x \sim y$ iff $x' \sim y'$ for any $x, y \in B_0$. Let $|x|, |y| \in B_0/\sim$. Then we put

$$|x| \leqslant |y| \quad \text{iff} \quad x = i_{\lambda_1} a,\ y = i_{\lambda_2} b \text{ and } a \leqslant b.$$

$\leqslant$ is a well-defined partial order in $B_0/\sim$. Let $|x| \in B_0/\sim$. Then we put

$$|x|' \overset{\text{df}}{=} |x'|.$$

The operation $|x| \mapsto |x|'$ is well defined.

We shall prove that the system

$$\mathscr{B}_0/\sim\ = \langle B_0/\sim; \leqslant, '\rangle$$

is a quantum logic.

Conditions (1), (2) and (4) are satisfied in an obvious way. We check condition (3). Let $|x_1|, |x_2|, \ldots$ be orthogonal in $B_0/\sim$. Let $x_n = i_{\lambda_n} a_n$. Then

$$|x_m| \perp |x_n| \quad \text{iff} \quad |x_m| \leqslant |x_n'| \quad \text{iff} \quad a_m \leqslant a_n' \quad \text{iff} \quad a_m \perp a_n.$$

Let $a = a_1 + a_2 + \ldots$ By Theorem 3 there exists a Boolean σ-algebra $\mathscr{A}_{\lambda_0}$ in $\mathscr{L}$ ($\lambda_0 \in \Lambda$) such that $a', a_1, a_2, \ldots$ belong to A_{λ_0}. Let $y_0 = i_{\lambda_0} a$ and $y_n = i_{\lambda_0} a_n$ ($n \in N$). Then $|x_n| = |y_n|$ for every $n \in N$. It is clear that y_0 is the l.u.b. of the layers $|y_1|, |y_2|, \ldots$ in $B_0/\sim$, i.e., $|y_0| = |y_1| + |y_2| + \ldots$

We show that condition (5) is satisfied in $\mathscr{B}_0/\sim$. Let $|x| \leqslant |y|$ and $x = i_{\lambda_1} a$, $y = i_{\lambda_2} b$. Then $a \leqslant b$. There exists a Boolean σ-algebra $\mathscr{A}_{\lambda_0}$ in $\mathscr{L}$ such that $a, b \in A_{\lambda_0}$. Let $x_0 = i_{\lambda_0} a$ and $y_0 = i_{\lambda_0} b$. Obviously, $|x| = |x_0|$ and $|y| = |y_0|$. Moreover, $x_0, y_0 \in B_{\lambda_0}$. Proving (3) we have shown that if $a_m \perp a_n$ ($m \neq n$) in L, $a = a_1 + a_2 + \ldots$ and $x_n = i_{\lambda_n} a_n$, $y = i_{\lambda_0} a$, then $|y| = |x_1| + |x_2| + \ldots$ It follows from $b' \perp a$ that

$$|y_0' \cup x_0| = |y_0|' + |x_0|.$$

Hence

$$|y_0| \cap |x_0|' = (|y_0|' + |x_0|)' = |y_0' \cup x_0|' = |y_0 \cap x_0'|.$$

As $a \perp b \cap a'$ and $b = a + (b \cap a')$, we obtain

$$|x_0| + (|y_0| \cap |x_0|') = |x_0| + |y_0 \cap x_0'| = |x_0 \cup (y_0 \cap x_0')| = |y_0|.$$

Therefore, the system $\mathscr{B}_0/\sim\ = \langle B_0/\sim; \leqslant, '\rangle$ is a quantum logic. Moreover, we have shown that $\mathscr{B}_0/\sim$ is isomorphic to $\mathscr{L}$. Thus we have

THEOREM 3. (i) *$\mathscr{B}_0/\sim = \langle B_0/\sim; \leqslant, '\rangle$ is a quantum logic.*

(ii) *The mapping $x \mapsto |x|$ is a homomorphism of the logic $\mathscr{B}_0$ onto $\mathscr{B}_0/\sim$.*

(iii) *$\mathscr{L}$ and $\mathscr{B}_0/\sim$ are isomorphic* (¹).

Now let us focus our attention on σ-orthodistributive quantum logics. After Mączyński [4] we accept the following definition:

Definition 3. A quantum logic $\mathscr{L} = \langle L; \leqslant, '\rangle$ is said to be *σ-orthodistributive* provided that for every (σ, σ)-indexed set $\{a_{n,j}\}_{n\in N, j\in N}$ of elements of L, where $\{a_{n,j}\}_{1\leqslant j<\infty}$ is orthogonal for every n, if

$$\bigcup_{j\in N} a_{n,j} = 1 \quad \text{for every } n \in N,$$

then for every $a \neq 0$ $(a \in L)$ there exist a mapping $\varphi \in N^N$ and $b \in L$ not orthogonal to a such that

$$b \leqslant a_{n,\varphi(n)} \quad \text{for all } n \in N.$$

In the case where $\mathscr{L}$ is a Boolean σ-algebra, σ-orthodistributivity of $\mathscr{L}$ coincides with σ-distributivity of $\mathscr{L}$ in the sense accepted in the theory of Boolean algebras.

Let $\mathscr{L}$ be a quantum logic and $a \in L$. We admit the notation $(+1)a = a$ and $(-1)a = a'$.

We shall use the following lemma which is a part of Theorem 1 in [4]:

LEMMA 1. *Let $\mathscr{L} = \langle L; \leqslant, '\rangle$ be a σ-orthodistributive quantum logic. Then $\mathscr{L}$ fulfills the following condition:*

If $\{a_n\}_{n\in N}$ is a σ-indexed set of elements in L, then for every element $a \in L$, $a \neq 0$, there exist a function $\varepsilon \in \{-1, +1\}^N$ and b not orthogonal to a such that $b \leqslant \varepsilon(n)a_n$ for all $n \in N$.

As before, let (6) be any Boolean σ-product of all σ-algebras $\mathscr{A}_\lambda$ $(\lambda \in \Lambda)$ contained in a quantum logic $\mathscr{L}$. Let Δ denote the Boolean symmetrical difference. Let $I_0 \subseteq B$ (B — the carrier of $\mathscr{B}$) be defined as follows:

$$I_0 = \{x\Delta y : x \sim y\}.$$

Notice that if $x_1\Delta y_1 = x_2\Delta y_2 \neq 0$, where $x_i \sim y_i$ $(i = 1, 2)$, then either $x_1 = x_2$, $y_1 = y_2$, or $x_1 = y_2$, $y_1 = x_2$. Moreover, for any $z \in I_0$, $z \neq 1$.

LEMMA 2. *Let $\mathscr{L}$ be a σ-orthodistributive quantum logic. Then, for every denumerable sequence $z_1, z_2, \ldots$ of elements of the set I_0 and for any fixed element x_0 belonging to the quantum logic $\mathscr{B}_0$ $(x_0 \neq 0)$,*

$$x_0 \cap \Big(\bigcup_{n\in N} z_n\Big)' \neq 0$$

in the algebra $\mathscr{B}$.

(¹) A similar theorem for partial Boolean σ-algebras has been given in [1].

Proof. Let $z_n = x_n \Delta y_n$, where $x_n \sim y_n$ $(n \in N)$. We may assume that $z_n \neq 0$ for every $n \in N$. Let $x_n = i_{\mu_n}(a_n)$, $y_n = i_{\nu_n}(a_n)$ for $n = 1, 2, \ldots$ $(\mu_n, \nu_n \in \Lambda)$ and $x_0 = i_{\lambda_0}(a_0)$. Then $a_n \neq 0, 1$ for every $n \in N$ and indices μ_n and ν_n are uniquely determined. By σ-orthodistributivity of $\mathscr{L}$, for the sequence $a_0, a_1, a_2, \ldots$ and the element a_0 one can choose a function $\varepsilon \in \{-1, +1\}^{N \cup \{0\}}$ and an element $b \in L$ not orthogonal to a and such that $b \leqslant \varepsilon(n) a_n$ for every $n \in N \cup \{0\}$. From b non $\perp a_0$ and $b \leqslant \varepsilon(0) a_0$ it follows that $\varepsilon(0) = +1$, i.e., $b \leqslant a_0$, and $b \neq 0$.

Let Λ_0 $(\Lambda_0 \subseteq \Lambda)$ be the set-theoretical sum of the set $\{\lambda_0\}$ and of the set of all elements in Λ occurring in the sequences $(\mu_1, \mu_2, \ldots)$ and $(\nu_1, \nu_2, \ldots)$. The set Λ_0 is countable. Thus for every $\lambda \in \Lambda_0$

$$b \leqslant b_\lambda \overset{\mathrm{df}}{=} \bigcap_{a_n \in \mathscr{A}_\lambda} \varepsilon(n) a_n,$$

where $\bigcap_{a_n \in \mathscr{A}_\lambda} \varepsilon(n) a_n$ denotes the g.l.b. of all elements of the sequence $(\varepsilon(0) a_0, \varepsilon(1) a_1, \varepsilon(2) a_2, \ldots)$ belonging to the σ-algebra $\mathscr{A}_\lambda$. Hence

$$i_\lambda(b_\lambda) = \bigcap_{a_n \in \mathscr{A}_\lambda} \varepsilon(n) i_\lambda(a_n) \neq 0$$

in the σ-algebra $\mathscr{B}_\lambda = i_\lambda(\mathscr{A}_\lambda) \subseteq \mathscr{B}$. It follows from σ-independency of the algebras $\{\mathscr{B}_\lambda\}_{\lambda \in \Lambda}$ that

$$\bigcap_{\lambda \in \Lambda_0} i_\lambda(b_\lambda) = \bigcap_{\lambda \in \Lambda_0} \bigcap_{a_n \in \mathscr{A}_\lambda} \varepsilon(n) i_\lambda(a_n) \neq 0$$

in the σ-algebra $\mathscr{B}$. But

$$\bigcap_{\lambda \in \Lambda_0} \bigcap_{a_n \in \mathscr{A}_\lambda} \varepsilon(n) i_\lambda(a_n) \leqslant i_{\lambda_0}(a_0) = x_0,$$

$$\bigcap_{\lambda \in \Lambda_0} \bigcap_{a_n \in \mathscr{A}_\lambda} \varepsilon(n) i_\lambda(a_n) \leqslant \varepsilon(n) i_{\mu_n}(a_n) = \varepsilon(n) x_n \qquad (n \in N),$$

$$\bigcap_{\lambda \in \Lambda_0} \bigcap_{a_n \in \mathscr{A}_\lambda} \varepsilon(n) i_\lambda(a_n) \leqslant \varepsilon(n) i_{\nu_n}(a_n) = \varepsilon(n) y_n \qquad (n \in N).$$

Hence

$$\bigcap_{\lambda \in \Lambda_0} \bigcap_{a_n \in \mathscr{A}_\lambda} \varepsilon(n) i_\lambda(a_n) \leqslant x_0 \cap \bigcap_{n=1}^{\infty} \big(\varepsilon(n) x_n \cap \varepsilon(n) y_n\big) \leqslant x_0 \cap \bigcap_{n=1}^{\infty} [(x_n \cap y_n) \cup (x_n' \cap y_n')]$$

$$= x_0 \cap \Big(\bigcup_{n=1}^{\infty} (x_n \Delta y_n)\Big)' = x_0 \cap \Big(\bigcup_{n=1}^{\infty} z_n\Big)'.$$

Thus

$$x_0 \cap \Big(\bigcup_{n=1}^{\infty} z_n\Big)' \neq 0.$$

It follows from Lemma 2 that I_0 generates a proper σ-ideal in the σ-algebra $\mathscr{B}$, which will be denoted by I. Moreover, if $x_0 \in B_0$, $x_0 \neq 0$, then $x_0 \notin I$. The ideal I determines a congruence $\approx$ in $\mathscr{B}$: $x \approx y$ iff $x \Delta y \in I$.

Let $\mathscr{A} = \mathscr{B}/I$ and let $\|x\|$ be a layer in $\mathscr{A}$ determined by x ($x \in B$). The mapping $x \mapsto \|x\|$ is a σ-homomorphism of $\mathscr{B}$ onto $\mathscr{A}$ (in the sense of the theory of Boolean algebras). Let $x, y \in B_0$. Then $x \sim y$ implies $x \Delta y \in I_0 \subseteq I$. Hence $x \approx y$.

The mapping $\psi: B_0/\sim \mapsto A$ determined by means of the formula $\psi(|x|) = \|x\|$ is well-defined. We claim that ψ is a homomorphism of the quantum logic $\mathscr{B}_0/\sim$ into a Boolean σ-algebra $\mathscr{A}$. It suffices to show that if $|x_1|, |x_2|, \ldots$ is an orthogonal sequence in $B_0/\sim$ and $|x| = |x_1| + |x_2| + \ldots$, then

$$\|x\| = \bigcup_{n=1}^{\infty} \|x_n\|$$

in the σ-algebra $\mathscr{A}$. It follows from Theorems 2 and 3 that there exist elements $y \in |x|$, $y_n \in |x_n|$ and a σ-algebra $\mathscr{B}_{\lambda_0}$ ($\lambda_0 \in \Lambda$) such that $y, x_n \in B_{\lambda_0}$ ($n \in N$). But $y = y_1 + y_2 + \ldots$ in the quantum logic $\mathscr{B}_0$. Hence

$$y = \bigcup_{n=1}^{\infty} y_n$$

in the σ-algebra $\mathscr{B}$. Then

$$\|y\| = \bigcup_{n=1}^{\infty} \|y_n\|$$

in the σ-algebra $\mathscr{A}$, i.e.,

$$\|x\| = \bigcup_{n=1}^{\infty} \|x_n\|.$$

We prove that ψ is a weak embedding, i.e., if $|x| \neq |y|$, $|x|$ and $|y|$ are compatible in $B_0/\sim$, then $\|x\| \neq \|y\|$ in the σ-algebra $\mathscr{A}$. It follows from Theorems 1 and 3 that there exist $\lambda \in \Lambda$ and $x_0 \in |x|$, $y_0 \in |y|$ such that $x_0, y_0 \in B_\lambda$ (B_λ — the carrier of $\mathscr{B}_\lambda$). Moreover, $x_0 \Delta y_0 \neq 0$ in B and $x_0 \Delta y_0 \in B_0$ (B_0 — the carrier of the quantum logic $\mathscr{B}_0$). By Lemma 2, $x_0 \Delta y_0 \notin I$. Hence

$$\|x_0\| \Delta \|y_0\| = \|x_0 \Delta y_0\| \neq 0$$

in the σ-algebra $\mathscr{A}$, which means that $\|x_0\| \neq \|y_0\|$. But $\|x\| = \|x_0\|$ and $\|y\| = \|y_0\|$. Hence $\|x\| \neq \|y\|$.

Thus we have proved the following

THEOREM 4. *Let $\mathscr{L}$ be a σ-orthodistributive quantum logic. Then $\mathscr{L}$ can weakly be embedded into a Boolean σ-algebra.*

Now, we intend to devote some space to complete quantum logics. We say that a quantum logic $\mathscr{L}$ is *complete* if, for every family $\{a_i\}_{i \in I}$ (I — any non-empty set) of pairwise orthogonal elements in L, the l.u.b. $\bigcup_{i \in I} a_i$ exists in L. Finch [3] uses the term *completely σ-orthomodular poset*.

Notice that if $\mathscr{L}$ is a quantum logic such that every orthogonal set in L is countable, then $\mathscr{L}$ is a complete quantum logic. Due to Finch [3] such a quantum logic is called *separable*.

LEMMA 3. *Let $\mathscr{L}$ be a σ-orthodistributive complete quantum logic. Let $a, b \in L$ and $a \operatorname{non}\leqslant b$. Then there exists an element $d \in L$, $d \neq 0$, such that $d \leqslant a$ and $d \leqslant b'$.*

Proof. We consider two cases.

I. a and b are compatible.

Then $a \cap b' \neq 0$. We put $d = a \cap b'$.

II. a and b are not compatible.

Suppose that there is no element d, $0 \neq d \in L$, such that $d \leqslant a$ and $d \leqslant b'$. First, we prove that then there is a $c \in L$, $c \neq 0$, such that $c \leqslant a$ and $c \leqslant b$. Indeed, by σ-orthodistributivity of $\mathscr{L}$ and Lemma 1, for the sequence (a, b) and the element a one can choose $\varepsilon, \delta \in \{-1, +1\}$ and $c \in L$ such that $c \operatorname{non}\perp a$, $c \leqslant \varepsilon a$ and $c \leqslant \delta b$. It follows that $c \neq 0$ and $\varepsilon = +1$, i.e., $c \leqslant a$. By our supposition one must have $\delta = +1$, i.e., $c \leqslant b$. Let

$$X_{a,b} \overset{\mathrm{df}}{=} \{c \in L\colon\ c \neq 0 \mathbin{\&} c \leqslant a \mathbin{\&} c \leqslant b\}.$$

Thus $X_{a,b}$ is non-empty. Let $M \subseteq X_{a,b}$ be any orthogonal set, maximal in $X_{a,b}$ with respect to the inclusion. By completeness of $\mathscr{L}$, $c_0 = \sup M$ exists and $c_0 \neq 0$. Then

$$c_0 \leqslant a,\ c_0 \leqslant b \quad \text{and} \quad a \cap c_0' \neq 0,\ b \cap c_0' \neq 0.$$

By σ-orthodistributivity of $\mathscr{L}$, for the sequence $(a \cap c_0', b \cap c_0')$ and the element $a \cap c_0'$ one can choose $\varepsilon, \delta \in \{-1, +1\}$ and an element $d \in L$ such that

$$d \operatorname{non}\perp a \cap c_0', \qquad d \leqslant \varepsilon(a \cap c_0'), \qquad d \leqslant \delta(b \cap c_0').$$

It follows that $d \neq 0$ and $\varepsilon = +1$, i.e., $d \leqslant a \cap c_0' \leqslant a$. We must have $\delta = +1$. Suppose, otherwise, that $\delta = -1$, i.e., $d \leqslant (b \cap c_0')' = b' \cup c_0$. But $d \leqslant a \cap c_0'$ implies $d \leqslant c_0'$, which together with $b' \perp c_0$ implies

$$(*) \qquad d \leqslant (b' \cup c_0) \cap c_0' = b'.$$

But $d \leqslant a \cap c_0 \leqslant a$ and $(*)$ are contradictory to our supposition. Thus we must have $\delta = +1$, which means that $d \leqslant b \cap c_0'$. The conditions $d \neq 0$, $d \leqslant a \cap c_0'$ and $d \leqslant b \cap c_0'$ imply $d \in X_{a,b}$ and $d \perp c_0$. This contradicts maximality of M.

Thus the supposition that there exists no element $d \in L$, $d \neq 0$, such that $d \leqslant a$ and $d \leqslant b'$ leads to a contradiction.

THEOREM 5. *Let $\mathscr{L}$ be a complete σ-orthodistributive quantum logic. Then $\mathscr{L}$ can be embedded into a Boolean algebra.*

The proof is carried on in the way similar to that used for Theorem 4. For convenience we preserve notation and definitions admitted there. First we prove

LEMMA 4. *Let $\mathscr{L}$ be a complete σ-orthodistributive quantum logic. Then for every denumerable sequence $z_1, z_2, \ldots$ of elements of the set I_0 and for any elements x and y belonging to the quantum logic $\mathscr{B}_0$ such that $x \operatorname{non} \sim y$ we have*

$$(x \Delta y) \cap \Big(\bigcup_{n \in N} z_n\Big)' \neq 0$$

in the algebra $\mathscr{B}$.

Proof. Let $z_n = x_n \Delta y_n$ $(x_n \sim y_n, n = 1, 2, \ldots)$ and $x_n = i_{\mu_n}(a_n)$, $y_n = i_{\nu_n}(a_n)$ for $n = 1, 2, \ldots$ $(\mu_n, \nu_n \in \Lambda)$. Let $x = i_{\lambda_1}(a_0)$ and $y = i_{\lambda_2}(b_0)$. Then $a_0 \neq b_0$. We consider two cases.

I. a_0 and b_0 are not compatible.

Then $a_0 \operatorname{non} \leqslant b_0$. By Lemma 3 there is a $d_0 \neq 0$ such that $d_0 \leqslant a_0$ and $d_0 \leqslant b_0'$. By Lemma 1, for the sequence $d_0, a_1, a_2, \ldots$ and the element d_0 one can choose a function $\varepsilon \in \{-1, +1\}^{N \cup \{0\}}$ and an element $e_0 \in L$ not orthogonal to d_0 such that $e_0 \leqslant \varepsilon(0) d_0$ and $e_0 \leqslant \varepsilon(n) a_n$ $(n = 1, 2, \ldots)$. Hence $e_0 \leqslant d_0$ and $e_0 \neq 0$. Thus e_0 is less than or equal to any element occurring in the sequence $\big(a_0, b_0', \varepsilon(1) a_1, \varepsilon(2) a_2, \ldots\big)$. It will be convenient to re-enumerate the elements of this sequence. Let

$$c_1 = a_0, c_2 = b_0, c_3 = a_1, c_4 = a_2, \ldots, c_k = a_{k-2}, \ldots \quad (k \geqslant 3).$$

Let $\delta \in \{-1, +1\}^N$ be defined as follows:

$$\delta(1) = +1, \delta(2) = -1, \delta(3) = \varepsilon(1), \ldots, \delta(k) = \varepsilon(k-2), \ldots \quad (k \geqslant 3).$$

Thus the sequences

$$\big(a_0, b_0', \varepsilon(1) a_1, \varepsilon(2) a_2, \ldots\big) \quad \text{and} \quad \big(\delta(1) c_1, \delta(2) c_2, \delta(3) c_3, \delta(4) c_4, \ldots\big)$$

are identical. Hence $e_0 \leqslant \delta(n) c_n$ for every $n \in N$.

Let Λ_0 be the set-theoretical sum of the set $\{\lambda_1, \lambda_2\}$ and of the set of all elements of Λ occurring in the sequences $(\mu_1, \mu_2, \ldots)$ and $(\nu_1, \nu_2, \ldots)$. The set Λ_0 is countable. Hence for every $\lambda \in \Lambda_0$

$$e_0 \leqslant d_\lambda \overset{\mathrm{df}}{=} \bigcap_{c_n \in \mathscr{A}_\lambda} \delta(n) c_n,$$

where $\bigcap_{c_n \in \mathscr{A}_\lambda} \delta(n) c_n$ denotes the g.l.b. of all elements of the sequence $\big(\delta(1) c_1, \delta(2) c_2, \ldots\big)$ belonging to the σ-algebra $\mathscr{A}_\lambda$. Hence

$$i_\lambda(d_\lambda) = \bigcap_{c_n \in \mathscr{A}_\lambda} \delta(n) i_\lambda(c_n) \neq 0$$

in the σ-algebra $\mathscr{B}_\lambda = i_\lambda(\mathscr{A}_\lambda) \subseteq \mathscr{B}$. From σ-independence of the algebras $\{\mathscr{B}_\lambda\}_{\lambda \in \Lambda}$ it follows that

$$\bigcap_{\lambda \in \Lambda_0} i_\lambda(d_\lambda) = \bigcap_{\lambda \in \Lambda_0} \bigcap_{c_n \in \mathscr{A}_\lambda} \delta(n) i_\lambda(c_n) \neq 0$$

in the σ-algebra $\mathscr{B}$. But

$$\bigcap_{\lambda\in\Lambda_0}\ \bigcap_{c_n\in\mathcal{A}_\lambda}\delta(n)i_\lambda(c_n)\leqslant\delta(1)i_{\lambda_1}(c_1)=x,$$

$$\bigcap_{\lambda\in\Lambda_0}\ \bigcap_{c_n\in\mathcal{A}_\lambda}\delta(n)i_\lambda(c_n)\leqslant\delta(2)i_{\lambda_2}(c_2)=y',$$

$$\bigcap_{\lambda\in\Lambda_0}\ \bigcap_{c_n\in\mathcal{A}_\lambda}\delta(n)i_\lambda(c_n)\leqslant\delta(k)i_{\mu_{k-2}}(c_k)=\varepsilon(k-2)x_{k-2}\qquad(k=3,4,\ldots),$$

$$\bigcap_{\lambda\in\Lambda_0}\ \bigcap_{c_n\in\mathcal{A}_\lambda}\delta(n)i_\lambda(c_n)\leqslant\delta(k)i_{\nu_{k-2}}(c_k)=\varepsilon(k-2)y_{k-2}\qquad(k=3,4,\ldots).$$

Hence

$$\begin{aligned}\bigcap_{\lambda\in\Lambda_0}i_\lambda(d_\lambda)&\leqslant x\cap y'\cap\bigcap_{n=1}^{\infty}\big(\varepsilon(n)x_n\cap\varepsilon(n)y_n\big)\\&\leqslant x\cap y'\cap\bigcap_{n=1}^{\infty}[(x_n\cap y_n)\cup(x_n'\cap y_n')]\leqslant(x\Delta y)\cap\Big(\bigcup_{n=1}^{\infty}(x_n\Delta y_n)\Big)'\\&=(x\Delta y)\cap\Big(\bigcup_{n=1}^{\infty}z_n\Big)'.\end{aligned}$$

Thus

$$(x\Delta y)\cap\Big(\bigcup_{n=1}^{\infty}z_n\Big)'\neq 0.$$

II. a_0 and b_0 are compatible.

Then either $a_0\cap b_0'\neq 0$ or $b_0\cap a_0'\neq 0$. Suppose that $a_0\cap b_0'\neq 0$. Let $d_0=a_0\cap b_0'$. By Lemma 1, for the sequence $d_0, a_1, a_2, \ldots$ and the element d_0 one may choose a function $\varepsilon\in\{-1,+1\}^{N\cup\{0\}}$ and an element $e_0\in L$ not orthogonal to d_0 such that $e_0\leqslant\varepsilon(0)d_0$ and $e_0\leqslant\varepsilon(n)a_n$ $(n=1,2,\ldots)$. Next we apply the identical reasoning as in case I.

Let $\approx$ be the congruence determined by the σ-ideal I in the algebra $\mathscr{B}$: $x\approx y$ iff $x\Delta y\in I$. It follows from Lemma 4 that if $x\,\mathrm{non}\sim y$, then $x\,\mathrm{non}\approx y$.

We prove that the mapping $\psi\colon B_0/\sim\ \mapsto A$ is one-to-one. Let $|x|\neq|y|$ in $B_0/\sim$. Thus $x\,\mathrm{non}\sim y$. Hence $x\,\mathrm{non}\approx y$, which means that $\|x\|\neq\|y\|$ in the algebra $\mathscr{A}$.

Thus Theorem 5 has been proved.

PROBLEM (**P 1064**). Can every σ-orthodistributive quantum logic be weakly embedded into a σ-distributive Boolean algebra?

Let $\mathscr{L}=\langle L;\leqslant,{}',1\rangle$ be any σ-orthomodular poset and, as before, let $\{\mathscr{A}_\lambda\colon\lambda\in\Lambda\}$ be the family of all Boolean σ-algebras in $\mathscr{L}$. Let $\lambda\leqslant\mu$ iff $A_\lambda\subseteq A_\mu$ and for $\lambda\leqslant\mu$ let $f_{\lambda,\mu}$ be the identity map from A_λ into A_μ $(\lambda,\mu\in\Lambda)$. Then

$$\{\mathscr{A}_\lambda;\ f_{\lambda,\mu}\colon\lambda,\mu\in\Lambda\}\tag{7}$$

forms a partially ordered system of Boolean algebras in $\boldsymbol{B}_{\aleph_0}$ in the sense of Dwinger (see [2], p. 320). Assume that (6) is a maximal Boolean σ-product. Then the Boolean σ-algebra $\mathscr{A} = \mathscr{B}/I$, considered above, coincides with the so-called direct limit of system (7) in $\boldsymbol{B}_{\aleph_0}$ (²). Hence Theorem 5 can be expressed as follows (see [2], p. 323):

THEOREM 6. *Suppose that* $\mathscr{L} = \langle L; \leqslant, ', 1\rangle$ *is a* σ*-orthodistributive* σ*-orthomodular poset. Let* (7) *be a partially ordered system in* $\boldsymbol{B}_{\aleph_0}$ *defined as above, whose direct limit* (*in* $\boldsymbol{B}_{\aleph_0}$) *is* $(\mathscr{A}, \{j_\lambda : \lambda \in \Lambda\})$. *Then, for every* $\lambda \in \Lambda$, j_λ *is a monomorphism of* $\mathscr{A}_\lambda$ *into* $\mathscr{A}$.

(²) The author wishes to thank Professor M. Mączyński for making this observation.

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