COLLOQUIUM MATHEMATICUM

VOL. XXIII 1971 FASC. 1

A FURTHER RESTRICTED ω-RULE

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In this paper* we extend a result of Shoenfield given in [6] (1). Namely, if we take the system Z_μ of [1] with definitions added for all primitive recursive functions, then the system Z_0^* is complete, where Z_0^* is obtained from Z_μ by adding the following restricted form of the ω-rule. $\forall x F(x)$ is to be a theorem of Z_0^* whenever there is a primitive recursive function φ such that, for each n, $\varphi(n)$ is a proof number of $F(\bar{n})$ in Z_0^*. The method of proof is analogous to [6]. Definitions of concepts undefined here are to be found in [6] and [2].

If B is a formula of Z_μ, then $[B]$ denotes the Gödel number of B. Let $\mathrm{Prf}(a, [B])$, $\mathrm{In}(e, n)$ be primitive recursive predicates expressing that a is a Gödel number of a proof in Z_μ of B, and e is an index for a primitive recursive function of n variables as in [3], respectively. The definition of a proof number in Z_0^* is given inductively as in [6], except that only e such that $\mathrm{In}(e, 1)$ are permitted as indices for functions in an application of the ω-rule. If $A(x, y)$ is a formula of Z_μ, we often write $x \leqslant_A y$ for $A(x, y)$. If $A(x, y)$ is quantifier-free, hence primitive recursive, and the sentence $\mathrm{Od}(\leqslant_A)$ of Z_μ, which expresses $x \leqslant_A y$ is a linear ordering, is provable in Z_μ, then $x \leqslant_A y$ is called an *ordering formula*. The following lemma is proved in § 6 of [4]:

LEMMA 1. *If B is a sentence of Z_μ, there is an ordering formula $x \leqslant_A y$ and a term $t(x)$ such that $(\exists x)\big(\neg\, t(x+1) <_A t(x)\big) \rightarrow B$ is provable in Z_μ. Moreover, if B is true, the predicate expressed by $x \leqslant_A y$ is a well-ordering.*

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* This paper is Part II of the author's Ph.D. thesis at Case Western Reserve University, written under the direction of Professor C. F. Kent to whom the author expresses his sincerest appreciation.

(1) The statement that formula (1) of [6], p. 406, is provable in Z_μ for arbitrary ordering formula $x \leqslant_A y$ and term $t(x)$ is incorrect, e.g., if $x \leqslant_A y \equiv y \leqslant x$ and $t(x) = x$, then (1) is false. This fact was pointed out to me by C. F. Kent and it is the purpose of this paper to provide a correct proof based upon Shoenfield's method.

Several other proofs of the analogous result have been given independently by Lopez-Escobar [5] and C. F. Kent using a semantic tableaux method for a restricted ω-rule in a Gentzen-type system for arithmetic.

LEMMA 2. *Let $t(x)$ be a term of Z_μ. Then the following sentence is deducible in Z_μ from* $\mathrm{Od}(\leqslant_A)$:

$$(1)\quad (z)\big(z \leqslant_A z \to \forall y \exists x\big(x > y \wedge \neg\, t(x+1) <_{A,z} t(x)\big)\big) \to \forall y \exists x\big(x > y \wedge \neg\, t(x+1) <_A t(x)\big),$$

where $x \leqslant_{A,z} y \equiv x \leqslant_A y \wedge y <_A z$, *and* $x <_{A,z} y \equiv x \leqslant_{A,z} y \wedge x \neq y$ (²).

Proof. We show that $\forall y \exists x\big(x \geqslant y \wedge \neg\, t(x+1) <_A t(x)\big)$ is deducible from the premise of (1) under assumption that $\mathrm{Od}(\leqslant_A)$ holds. If $\neg\, t(y) \leqslant_A t(y)$, it is clear. If $t(y) \leqslant_A t(y)$, set $w = \mu x\big(x > y \wedge \neg\, t(x+1) <_{A,t(y)} t(x)\big)$, and hence either $\neg\, t(w+1) <_A t(w)$ or $\neg\, t(w) <_A t(y)$. If $\neg\, t(w) <_A t(y)$, then set $z = \mu x(x+1 = w)$ and show $z = y$. From here the result follows.

Let $x \leqslant_A y$ be a fixed ordering sentence of Z_μ. Since, for each a, b, either $\vdash_{Z_\mu} A(\bar a, \bar b)$ or $\vdash_{Z_\mu} \neg A(\bar a, \bar b)$, and $A(x, y)$ is provably equivalent to a primitive recursive predicate in Z_μ, there is an e such that $\mathrm{In}(e, 2)$ and, for every a, b,

$$(2)\quad \begin{array}{l} \mathrm{Prf}\big((\{e\}(a,b))_0, [A(\bar a, \bar b)]\big) \ \text{ iff } \ \big(\{e\}(a,b)\big)_1 = 0, \\ \mathrm{Prf}\big((\{e\}(a,b))_0, [\neg A(\bar a, \bar b)]\big) \ \text{ iff } \ \big(\{e\}(a,b)\big)_1 \neq 0. \end{array}$$

Moreover, it is possible uniformly from e to find such indices for the ordering sentences $\leqslant_{A,\bar n}$, as follows. There is an m, $\mathrm{In}(m, 3)$, such that: whenever $k = [A(x, y)]$, $\mathrm{In}(e, 2)$, and (2), then for each n, $\mathrm{In}(\{m\}(k, e, n), 2)$ and for all a, b, (2) holds with respect to $x \leqslant_{A,\bar n} y$ and $\{m\}(k, e, n)$.

Similarly, there is an r, $\mathrm{In}(r, 3)$, such that: whenever $k = [x \leqslant_A y]$ and $\mathrm{Prf}\big(a, [\mathrm{Od}(\leqslant_A)]\big)$, then for each n, $\mathrm{Prf}\big(\{r\}(k, a, n), [\mathrm{Od}(\leqslant_{A,\bar n})]\big)$. Finally, let b be such that $\mathrm{In}(b, 2)$ and $\{b\}([x \leqslant_A y], n) = [x \leqslant_{A,\bar n} y]$.

LEMMA 3. *Let $t(x)$ be a term of Z_μ. There is a g, $\mathrm{In}(g, 4)$, such that if:* (i) $k = [x \leqslant_A y]$, (ii) $\mathrm{Prf}\big(a, [\mathrm{Od}(\leqslant_A)]\big)$, (iii) $\mathrm{In}(e, 2)$, (iv) $\mathrm{In}(v, 1)$, (v) $A(x, y)$, *e satisfy* (2), *and* (vi) *if $n \leqslant_A n$, then $\{v\}(n)$ is a proof number in Z_0^* of* $\bar n \leqslant_A \bar n \to \forall y \exists x\big(x > y \wedge \neg t(x+1) <_{A,\bar n} t(x)\big)$. Then $\{g\}(k, a, e, v)$ is a proof number in Z_0^* of

$$(3)\quad \forall y \exists x\big(x > y \wedge \neg t(x+1) <_A t(x)\big).$$

Proof. From (ii) and Lemma 2, we have a proof of (1) in Z_μ. Now, we give a proof of the premise of (1) in Z_0^* and hence obtain a proof of (3) in Z_0^*. If $n \leqslant_A n\big((\{e\}(n, n))_1 = 0\big)$, then $\{v\}(n)$ is a proof number of $\bar n \leqslant_A \bar n \to \forall y \exists x\big(x > y \wedge \neg t(x+1) <_{A,\bar n} t(x)\big)$. If not $n \leqslant_A n\big((\{e\}(n, n))_1 \neq 0\big)$, then $\big(\{e\}(n, n)\big)_0$ is a proof number for $\neg(\bar n \leqslant_A \bar n)$ and hence we

(²) If one replaces (1) of [6] by formula (1) here, then Shoenfield's proof of the completeness of Z^* with recursive ω-rule is valid.

easily obtain a proof number for $\bar{n} \leqslant_A \bar{n} \rightarrow \forall y \exists x \big(x > y \wedge \neg t(x+1) <_{A,\bar{n}} t(x)\big)$. It is now easy to find an index i of a function primitive recursive uniformly in e, v which is a proof in Z_0^* of the premise of (1), and the definition of g follows routinely.

THEOREM. *If B is a true sentence of Z_μ, then B is provable in Z_0^*.*

Proof. By Lemma 1, it is sufficient to prove (3) for a suitable term $t(x)$ and ordering formula $x \leqslant_A y$ (which is a well-ordering). Moreover, there exists an e, $(\mathrm{In}(e, 2))$, such that (2) holds for $x \leqslant_A y$ and e. Let a be a number such that $\mathrm{Prf}\big(a, [\mathrm{Od}(\leqslant_A)]\big)$ and let $k = [x \leqslant_A y]$. Let s be an index such that $\mathrm{In}(s, 4)$ and

$$\{s\}(z, k, a, e) = \begin{cases} \{g\}\big(k, a, e, \lambda n \{z\}\big(\{b\}(k, n), \{r\}(k, a, n), \{m\}(k, e, n)\big)\big) & \text{if } \mathrm{In}(z, 3), \\ 0 & \text{if not } \mathrm{In}(z, 3). \end{cases}$$

The number g is given by Lemma 3 and b, r, m are to be as in the remarks preceding Lemma 3. By Kleene's recursion theorem for primitive recursive functions [3], p. 75, there is an index t_0 such that $\mathrm{In}(t_0, 3)$ and for all k, a, e, $\{t_0\}(k, a, e) = \{s\}(t_0, k, a, e)$. Now one readily proves by ordinal induction on the order type of $x \leqslant_A y$ that $\{t_0\}(k, a, e)$ is a proof number in Z_0^* of (3) where k, a, e, are as above.

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Reçu par la Rédaction le 6. 3. 1970