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CHARACTERIZING THE INTERVAL AND THE CIRCLE BY COMPOSITION OF FUNCTIONS

BY

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1. Introduction. Let X be a set and G a collection of endomorphisms of X which is closed under the operation of composition. We can put a quasi-ordering on G by setting $f < g$ when there exists $a \in G$ such that $fa = g$; this is the *intrinsic quasi-order* for G. A question which has arisen in mountain climbing [6], geometry [7], and universal mapping problems [1] and [4], is whether G is a directed set in this quasi-order. (Recall that G is directed if, for every $f, g \in G$, there is an $h \in G$ with $f < h$ and $g < h$; in the present setting, this means there are $\alpha, \beta \in G$ such that $f\alpha = g\beta$.)

Let X be a compact metric space, and let $C(X)$ be the collection of continuous maps taking X onto X with the intrinsic quasi-order and the metric induced by the sup norm. Recall that a Peano continuum is a connected, locally connected compact metric space.

THEOREM 1. *A non-degenerate Peano continuum X is topologically an interval if and only if $C(X)$ contains a dense directed subset.*

The "only if" statement of Theorem 1 is proved in [5], and examples showing that $C([0, 1])$ is not itself directed are presented in [3] and [6]. An interesting question is whether $C(X)$ is directed if X is the pseudo-arc (see [4] for definition). (**P 771**)

Let Y be an oriented PL manifold without boundary, and let $M(Y)$ be the subset of $C(Y)$ consisting of PL maps of degree one.

THEOREM 2. *$M(Y)$ is directed in its intrinsic quasi-order if and only if Y is topologically a circle.*

The "if" statement of Theorem 2 is Theorem 3.3 of [1].

2. Preliminaries. Our principal tool will be the double graph of a pair of maps, which was introduced in [3] and [6]. If X_1 and X_2 are topolo-

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gical spaces, let $C(X_1, X_2)$ be the collection of continuous maps taking X_1 onto X_2. For $f, g \in C(X_1, X_2)$ define the maps $f\times g\colon X_1\times X_1 \to X_2\times X_2$ by $f\times g(x, y) = \big(f(x), g(y)\big)$, and let $\Delta_X = \{(x, x)\colon x\in X\} \subset X\times X$; then the *double graph* of f and g is the subset $[f, g] = (f\times g)^{-1}(\Delta_{X_2})$ of $X_1\times X_1$. We shall use the following properties of double graphs, the first of which follows immediately from the definition.

(1) *Let $f, g, \alpha, \beta \in C(X)$; then $f\alpha = g\beta$ if and only if $\alpha\times\beta(\Delta_X) \subset [f, g]$.*

(2) *If $f, g \in C(X)$ and U is an open neighborhood of $[f, g]$ in $X\times X$, then there exist neighborhoods V_f of f and V_g of g in $C(X)$ such that $[f', g'] \subset U$ for every $f' \in V_f$ and $g' \in V_g$.*

Proof. Note that $f\times g(X\times X - U)$ is a compact set missing Δ_X; if f', g' are close f, g, then $f'\times g'(X\times X - U)$ also misses Δ_X, which means that $[f', g'] = (f'\times g')^{-1}(\Delta_X) \subset U$.

Let π_1 and π_2 be the projections of $X\times X$ onto its first and second factors. We will say that a set $K \subset X\times X$ is *full* if $\pi_1(K) = \pi_2(K) = X$. It is immediate that

(3) *if $f, g, \alpha, \beta \in C(X)$ and $f\alpha = g\beta$, then some component of $[f, g]$ is full.*

In the next proposition, T^i is the i-skeleton of a simplicial complex T.

(4) *If Y is an n-manifold, T is a triangulation of Y, and $f, g \in C(Y)$ are linear on simplexes of T and in general position, then $[f, g] - (|T^{n-1}| \times |T^{n-1}|)$ is an n-manifold.*

By the general position of f and g,

$$\dim\{[f, g]\cap(|T^{n-1}|\times|T^{n-1}|)\} \leqslant n-2;$$

thus $[f, g]$ is a pseudomanifold.

Proof. Since $[f, id_Y]$ is just the graph of f, it is clear that $[f, g]$ is an n-manifold if g is injective. Thus $[f, g] - (Y\times|T^{n-1}|)$ is an n-manifold for arbitrary g, and the conclusion follows from symmetry.

We will denote the maximum open subset of $[f, g]$ which is an n-manifold by $[f, g]^*$.

3. Proof of Theorem 1.

LEMMA. *Let X be a non-degenerate Peano continuum which is not homeomorphic to $[0, 1]$. Then there are maps $f, g \in C([0, 1], X)$ such that no component of $[f, g]$ is full.*

Proof. Since X is not homeomorphic to $[0, 1]$, it has at least 3 distinct non-cut points [2], which we will call a_1, a_2, and a_3. For any $f\in C([0, 1], X)$ we define $A_i(f) = f^{-1}(a_i)$, and we say that two points of $\bigcup_i A_i(f)$ are *adjacent* (with respect to f) if there is no point of $\bigcup_i A_i(f)$ between them on $[0, 1]$. We will construct $f, g \in C([0, 1], X)$ such that some point of

$A_1(f)$ is adjacent (w. r. t. f) to a point of $A_2(f)$ and no point of $A_1(g)$ is adjacent (w. r. t. g) to a point of $A_2(g)$.

Suppose for the moment that we have such a pair f, g, and that some component K of $[f, g]$ is full. Let $x_1 \in A_1(f)$ and $x_2 \in A_2(f)$ be adjacent (w. r. t. f), and suppose, for convenience, that $x_1 < x_2$. It follows from the fact that K is connected and full that $\pi_1(L) = [x_1, x_2]$ for some component L of $K \cap \pi_1^{-1}([x_1, x_2])$. If (x_1, y_1) and (x_2, y_2) belong to L, then $y_1 \in A_1(g)$ and $y_2 \in A_2(g)$, since $L \subset [f, g]$. By the choice of g, it follows that there is a point $y_3 \in A_3(g)$ between y_1 and y_2; since L is connected there is a point $(x_3, y_3) \in L$. But since $L \subset [f, g] \cap \pi_1^{-1}([x_1, x_2])$, this means that $x_3 \in A_3(f)$ is between x_1 and x_2, contradicting our choice of x_1 and x_2. Therefore, no component of $[f, g]$ is full.

To construct g, take a Cantor set T in $[0, 1]$ and a map $\varphi \in C([0, 1], X)$ such that $\varphi(T) = X$. It follows from the continuity of φ that there are only a finite number $n(\varphi)$ of pairs of points x_1, x_2 such that $x_1 \in A_1(\varphi)$, $x_2 \in A_2(\varphi)$ and x_1 is adjacent (w. r. t. φ) to x_2. Fix such a pair x_1, x_2, supposing for convenience that $x_1 < x_2$. Let (x_1', x_2') be an open interval in $[x_1, x_2] - T$, and let x be a point in (x_1', x_2'). Since $X - \{a_1\}$ and $X - \{a_2\}$ are connected, they are arc-connected [2], and we can find maps φ_1: $[x_1', x] \to X - \{a_2\}$ and φ_2: $[x, x_2'] \to X - \{a_1\}$ such that $\varphi_i(x_i') = \varphi(x_i')$ and $\varphi_i(x) = a_3$ $(i = 1, 2)$. Defining φ': $[0, 1] \to X$ by

$$\varphi'(t) = \begin{cases} \varphi(t) & \text{for } t \notin [x_1', x_2'], \\ \varphi_1(t) & \text{for } t \in [x_1', x], \\ \varphi_2(t) & \text{for } t \in [x, x_2'] \end{cases}$$

gives us a map $\varphi' \in C([0, 1], X)$ with $\varphi'(T) = X$ and $n(\varphi') < n(\varphi)$, and we induct on $n(\varphi)$ to get g. The map f is obtained in a single step by the process above.

Proof of Theorem 1. Let X be a non-degenerate Peano continuum which is not homeomorphic to [0,1]. The lemma gives us maps f^*, $g^* \in C([0, 1], X)$ such that no component of $[f^*, g^*]$ is full, and we set $f = f^*\varphi$ and $g = g^*\varphi$, where φ is any map of X onto $[0, 1]$. From the observation that $\varphi \times \varphi([f, g]) \subset [f^*, g^*]$ and $\varphi\pi_i = \pi_i(\varphi \times \varphi)$, it follows that no component of $[f, g]$ is full. We will now show that there are neighborhoods V_f of f and V_g of g in $C(X)$ such that no component of $[f', g']$ is full if $f' \in V_f$ and $g' \in V_g$. By (3), this suffices to prove the theorem.

Suppose that the neighborhoods described above do not exist. Then there are sequences $f_i \to f$ and $g_i \to g$ in $C(X)$ such that some component K_i of $[f_i, g_i]$ is full. We can pick a subsequence $\{K_{i_n}\}$ of $\{K_i\}$ such that $\liminf K_{i_n} = \limsup K_{i_n} = K$ ([2]). The limit set K is connected and full, and it follows from (2) that $K \subset [f, g]$, which is impossible by our choice of f and g.

4. Proof of Theorem 2. Suppose Y is a triangulated n-manifold, $n \geqslant 2$, and let Σ be an n-simplex in the triangulation of Y. The simplex Σ is PL homeomorphic whith the n-dimensional cube I^n, and for convenience we will identify Σ with I^n in our construction. Represent I^n as a product $I^{n-1} \times [0, 1]$, choose a point a in $\operatorname{int} I^{n-1}$, and set $a_i = (a, i/8) \in I^n$, $i = 1, \ldots, 7$. We define $f \in C(Y)$ in the following manner: $f(x) = x$ for $x \in Y - \operatorname{int}(I^{n-1} \times [0, 3/8]), f(a_1) = a_7$, and f is linear on segments $a_1 x$ for $x \in \partial(I^{n-1} \times [0, 3/8])$. Similarly, g is defined by $g(x) = x$ for $x \in Y - - \operatorname{int}(I^{n-1} \times [5/8, 1])$, $g(a_7) = a_1$, and g is linear on segments $a_7 x$ for $x \in \partial(I^{n-1} \times [5/8, 1]$.

The maps f and g are clearly in $M(Y)$, and are in general position. Let K be the component of $[f, g]$ containing Δ_{Y-I^n}. K contains an $(n-1)$-sphere $S = \Delta_{\partial(I^{n-1} \times [0, 1/2])}$ and a simple closed curve $I = (a_3, a_3)(a_5, a_5)(a_3, a_6)(a_2, a_5)(a_3, a_3)$ which intersect transversally at the single point (a_4, a_4). It is easily seen that $S \cup J \subset [f, g]^*$, and it follows from this that $H_{n-1}(K, Z_2)$ contains $k+1$ linearly independent elements with representations lying in $[f, g]^*$, where k is the number of elements in a basis for $H_{n-1}(\Delta_Y, Z_2)$. There is, therefore, no map of Δ_Y onto K of degree one, and thus no map φ: $\Delta_Y \to K$ with $\deg \pi_i \varphi = 1$; the theorem follows from (1).

Let $C'(Y)$ be the component of $C(Y)$ containing the identity map. The set $C'(Y)$ is closed under the operation of composition, and we consider it with its intrinsic quasi-order.

COROLLARY. *An oriented PL manifold Y is topologically a circle if and only if $C'(Y)$ contains a dense directed subset.*

Proof. This follows from (2) and the observation that $[f, g]$ has a regular neighborhood, where f and g are the functions constructed above.

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