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A UNIVERSAL CONVEX SET IN EUCLIDEAN SPACE

BY

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In 1935 S. Mazur posed the problem whether there exists a symmetric convex body Q in $\boldsymbol{R}^3$ such that every symmetric convex body in $\boldsymbol{R}^2$ is affinely equivalent to the intersection of Q with some 2-dimensional subspace (Problem 41 of The Scottish Book). This problem was solved in the negative by Grünbaum [3] (see also Bessaga [1]).

Professor C. Ryll-Nardzewski has asked a related question whether there exists a compact convex set Q in $\boldsymbol{R}^3$ such that every convex body in $\boldsymbol{R}^2$ is affinely isomorphic to the intersection of Q with some plane.

In this note we present an example of a compact convex set Q in $\boldsymbol{R}^{n+2}$ ($n \geqslant 1$) such that every closed convex subset of the unit ball B of $\boldsymbol{R}^n$ is obtained as an intersection of Q with some n-dimensional hyperplane of $\boldsymbol{R}^{n+2}$.

Let 2^B denote the space of all closed non-empty subsets of B endowed with the Hausdorff distance

$$\operatorname{dist}(A_1, A_2) = \max\big(\sup_{x \in A_1} d(x, A_2), \sup_{y \in A_2} d(y, A_1)\big),$$

where d stands for the Euclidean metric $d(x, y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}$ in $\boldsymbol{R}^n$. It is well known that 2^B is compact. It is also easy to see that if $\operatorname{dist}(A_n, A_0) \to 0$ and $d(x_n, x_0) \to 0$ as $n \to \infty$ with $x_n \in A_n \in 2^B$, then $x_0 \in A_0$.

LEMMA. *The set $\mathscr{C}$ of all convex sets in 2^B is a locally arcwise connected metric continuum.*

Proof. Let a sequence A_k of elements in $\mathscr{C}$ converge to $A_0 \in 2^B$ and suppose $x \in A_0$. Then, clearly, there exists a sequence (x_k) with $x_k \in A_k$ converging to x. This implies that if $x, y \in A_0$, then $rx + (1-r)y \in A_0$ for every r ($0 \leqslant r \leqslant 1$), so A_0 is convex, proving the compactness of $\mathscr{C}$.

Now we prove that $\mathscr{C}$ is locally arcwise connected. It is sufficient to show that for any two distinct $A_0, A_1 \in \mathscr{C}$ there exists an arc A_0A_1 with diameter less than or equal to $\operatorname{dist}(A_0, A_1)$ (see [2], p. 242). We put

$$A_t = tA_1 + (1-t)A_0 = \{ty + (1-t)x : x \in A_0, y \in A_1\} \in \mathscr{C}.$$

Let $x \in A_0$, $y \in A_1$ and let $x_0 \in A_0$, $y_0 \in A_1$ be such that

$$d(x, y_0) \leqslant \operatorname{dist}(A_0, A_1) \quad \text{and} \quad d(y, x_0) \leqslant \operatorname{dist}(A_0, A_1).$$

For $0 \leqslant t < s \leqslant 1$ we have

$$d\big(sy+(1-s)x, A_t\big) \leqslant d\left(sy+(1-s)x, ty+(1-t)\left[\frac{1-s}{1-t}x+\frac{s-t}{1-t}x_0\right]\right)$$
$$= \|(s-t)(y-x_0)\| \leqslant |s-t| \operatorname{dist}(A_0, A_1)$$

and, analogously,

$$d\big(ty+(1-t)x, A_s\big) \leqslant |s-t| \operatorname{dist}(A_0, A_1).$$

Thus for $t, s \in [0, 1]$ we have

$$\operatorname{dist}(A_t, A_s) \leqslant |s-t| \operatorname{dist}(A_0, A_1). \tag{1}$$

Let $x_1, x_2 \in A_0$ and $y_1, y_2 \in A_1$ be such that

$$\sup_{y \in A_1} d(y, A_0) = d(y_1, A_0) = d(y_1, x_1)$$

and

$$\sup_{x \in A_0} d(x, A_1) = d(x_2, A_1) = d(x_2, y_2).$$

Then

$$\sup_{x \in A_0, y \in A_1} d\big(ty+(1-t)x, A_0\big) \geqslant d\big(ty_1+(1-t)x_1, A_0\big) = td(x_1, y_1).$$

For any $y \in A_1$ we have $\big\|\big(ry+(1-r)y_2\big)-x_2\big\| \geqslant \|y_2-x_2\|$ for every $r \in [0, 1]$, so

$$\langle y-y_2, y_2-x_2\rangle \geqslant 0. \tag{2}$$

For any $x \in A_0$ there exists $y_3 \in A_1$ such that $d(x, y_3) \leqslant d(x_2, y_2)$. Now

$$\|(y_3-y_2)+(y_2-x_2)+(x_2-x)\|^2 \leqslant \|y_2-x_2\|^2,$$

and so

$$\|y_3-y_2+x_2-x\|^2+2\langle y_3-y_2, y_2-x_2\rangle \leqslant 2\langle y_2-x_2, x-x_2\rangle.$$

Since $\langle y_3-y_2, y_2-x_2\rangle \geqslant 0$, we have

$$\langle y_2-x_2, x-x_2\rangle \geqslant 0. \tag{3}$$

By (2) and (3),

$$\sup_{z \in A_0} d(z, A_t) \geqslant d(x_2, A_t)$$
$$= \inf_{x \in A_0, y \in A_1} \|t(y-y_2)+(1-t)(x-x_2)+t(y_2-x_2)\| \geqslant t\|y_2-x_2\|$$

and

$$\operatorname{dist}(A_0, A_1) \geqslant \max\big(td(x_1, y_1), td(x_2, y_2)\big) = t\operatorname{dist}(A_0, A_1),$$

so

$$\operatorname{dist}(A_0, A_t) = t\operatorname{dist}(A_0, A_1)$$

and, analogously,

$$\operatorname{dist}(A_t, A_1) = (1-t)\operatorname{dist}(A_0, A_1).$$

Therefore, by (1), for any $s, t \in [0, 1]$ we obtain

$$\operatorname{dist}(A_s, A_t) = |s-t|\operatorname{dist}(A_0, A_1),$$

so the arc $A_0A_1 = \{A_t\colon 0 \leqslant t \leqslant 1\}$ has diameter less than or equal to $\operatorname{dist}(A_0, A_1)$.

THEOREM. *For every $n \geqslant 1$ there exists a compact convex set Q in $\boldsymbol{R}^{n+2}$ such that every closed subset of the unit ball B of $\boldsymbol{R}^n$ can be obtained as an intersection of Q with some n-dimensional affine subspace of $\boldsymbol{R}^{n+2}$.*

Proof. By the Lemma and the Peano theorem ([2], p. 246), there exists a continuous function ψ from the interval $[0, 1]$ onto $\mathscr{C}$. For $t \in [0, 1]$ we write

$$C_t = \psi(t) \times \{(\cos t, \sin t)\} \subset \boldsymbol{R}^{n+2}$$

and put

$$Q = \operatorname{conv} \bigcup_{t\in[0,1]} C_t.$$

The set Q is compact. Indeed, let

$$\boldsymbol{x}_k = (x_k^1, \ldots, x_k^n, \cos t_k, \sin t_k) \in Q.$$

Since $\|\boldsymbol{x}_k\| \leqslant \sqrt{2}$, there exists a subsequence $\boldsymbol{x}_{k'}$ of $\boldsymbol{x}_k$ converging to some element

$$\boldsymbol{x}_0 = (x_0^1, \ldots, x_0^n, \cos t_0, \sin t_0).$$

Obviously, $t_{k'} \to t_0$ and $\boldsymbol{y}_{k'} = (x_{k'}^1, \ldots, x_{k'}^n) \to \boldsymbol{y}_0 = (x_0^1, \ldots, x_0^n)$ in $\boldsymbol{R}^n$. We have $\boldsymbol{y}_{k'} \in \psi(t_{k'})$ and $\operatorname{dist}(\psi(t_{k'}), \psi(t_0)) \to 0$. By the remark preceding the Lemma this implies that $\boldsymbol{y}_0 \in \psi(t_0)$, so $\boldsymbol{x}_0 \in Q$.

Since ψ is an onto mapping, for every convex subset D of B there exists $t \in [0, 1]$ such that $\psi(t) = D$ and for the n-dimensional affine subspace $H_t = \boldsymbol{R}^n \times \{(\cos t, \sin t)\}$ we have

$$Q \cap H_t = D \times \{(\cos t, \sin t)\}.$$

Indeed, if $\boldsymbol{x} \in Q \cap H_t$, then there exist elements $\boldsymbol{x}_i \in C_{t_i}$ and positive real numbers a_i $(i = 1, \ldots, m)$ such that $\sum a_i = 1$ and $\boldsymbol{x} = \sum a_i \boldsymbol{x}_i$. In particular,

$$\sum a_i(\cos t_i, \sin t_i) = (\cos t, \sin t).$$

By the strict convexity of the unit disc in $\boldsymbol{R}^2$ this implies $(\cos t_i, \sin t_i) = (\cos t, \sin t)$, i.e. $t_i = t$ for $i = 1, \ldots, m$. Thus $\boldsymbol{x} \in C_t \times (\cos t, \sin t)$. Since the reverse inclusion is obvious, the proof is complete.

Let us note that, by an easy application of the Peano theorem together with some of the arguments above, the (non-convex) set

$$P = \bigcup_{t\in[0,1]} \{(x_1, x_2, t): (x_1, x_2) \in \psi(t)\} \subset \mathbf{R}^3$$

satisfies the following condition:

Every closed convex set in $\mathbf{R}^2$ with diameter less than or equal to 1 can be obtained as the intersection of P with some plane.

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