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A PROPERTY OF THE LATTICE OF SUBSEMILATTICES

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We prove here that every infinite semilattice can be represented as a union of a strictly increasing sequence of subsemilattices. Thus, in the terminology of [2], we prove that infinite semilattices have the cofinality ω. Cofinality of groups was studied in [1], [3] and [4], and cofinality of Boolean algebras in [2]. Related theorems can be found in papers on Jónsson algebras.

It seems natural also to consider the dual problem, i.e., given an algebra A, is there a strictly decreasing sequence of congruences of A such that the intersection of this sequence is the smallest congruence of A? (**P 1084**)

THEOREM. *Infinite semilattices have the cofinality* ω.

Proof. Suppose first that $S = (S; \vee)$ contains an infinite decreasing sequence $s_0 > s_1 > \ldots$ Let

$$S_n = \{x \in S\colon x < s_n \to \mathop{\forall}_{k>0} x \leqslant s_{n+k}\}, \qquad n = 0, 1, \ldots$$

Let x and y be elements of S_n and suppose that $x \vee y < s_n$. Then $x < s_n$, $y < s_n$, and thus for every $k > 0$ we have $x \leqslant s_{n+k}$ and $y \leqslant s_{n+k}$. Thus $x \vee y \leqslant s_{n+k}$ and, accordingly, S_n is a subsemilattice. Suppose that $x \in S_n$ and $x < s_{n+1}$. Then $x < s_n$ and, consequently, $x \leqslant s_{n+k}$ for every $k > 0$, and so $S_n \subseteq S_{n+1}$. The sequence S_n is strictly increasing since $s_{n+1} \in S_{n+1} \setminus S_n$.

Finally, if $x \notin S_0$, then there is a k such that $x \nleqslant s_k$, but this implies that $x \in S_k$, and thus S is the union of the sequence S_n.

Thus we can assume that every decreasing sequence of elements of S is finite. Let $r(x)$ be the supremum of lengths of sequences of the form $x = x_0 > x_1 > \ldots > x_n$. Let F be the set of x such that $r(x)$ is finite. Clearly, F is infinite. Let $\sim$ be the equivalence relation such that $x \sim y$ if and only if $x = y$ or $r(x) = r(y) = \omega$. Obviously, $\sim$ is a congruence relation, and thus S has a homomorphism onto $S/\sim$. Since the inverse image of a subalgebra is a subalgebra and $S/\sim$ is infinite, it is enough to show that $S/\sim$ is a union of an increasing sequence of subsemilattices. Thus without loss of generality we can assume that for every element x of S, save possibly the largest element, $r(x)$ is finite.

Let $|X|$ denote the cardinality of the set X and let $|S| = \mathfrak{m}$. Let $R_n = \{x \in S: r(x) = n\}$, $n = 0, 1, \ldots$ Since the semilattice generated by an infinite set of cardinality $\mathfrak{l}$ has the cardinality $\mathfrak{l}$, we can assume that there is an n such that $|R_n| = \mathfrak{m}$, otherwise S is, trivially, a union of an increasing sequence of subsemilattices.

Let p be the smallest integer such that $|R_p| = \mathfrak{m}$. Notice that each of the sets R_n is an antichain, and thus we can find a maximal antichain M containing R_p. Clearly, if $x \in M$, then $r(x) \leqslant p$.

For $X \subseteq S$ we put

$$X^* = \{x \in S: \underset{y \in X}{\exists}\ x \geqslant y\}.$$

Since p is the smallest integer such that $|R_p| = \mathfrak{m}$, we have $|S \setminus M^*| < \mathfrak{m}$. Let L be the subsemilattice generated by $S \setminus M^*$. Then $|L| < \mathfrak{m}$. Let M_n be a strictly increasing sequence of subsets of M such that

$$M = \bigcup_{n \in \omega} M_n \quad \text{and} \quad M_0 = M \cap L.$$

Let T_n be the subsemilattice generated by $M_n^* \cup L$. Since M is an antichain and, for every $x \in M$, $r(x) = p$ if and only if $p \in R_p$, we infer that the sequence T_n is strictly increasing. Since M is maximal, we have

$$\bigcup_{n \in \omega} T_n = S.$$

Thus the Theorem is proved.

REFERENCES

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[4] G. Sabbagh, *Sur les groupes qui ne sont pas réunion d'une suite croissante de sous-groupes propres*, ibidem 280 (1974), p. 763-766.

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