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GENERALIZED ALMOST PERIODIC FUNCTIONS

BY

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1. Introduction. The almost periodic functions of Bohr have been generalized by, among others, von Neumann, Bochner, Stepanoff, Weyl, Besicovitch, Maak, and more recently, de Leeuw and Glicksberg. The following paper (1) is intended to clarify the interplay between these various ideas, as well as to extend them somewhat.

Definition 0. Let L be a locally convex, complete (closure of each totally bounded set is compact) linear topological space with neighborhood basis $\mathscr{U}$ at 0. For any set X, $B(X, L)$ is the set of all functions $f\colon X \to L$ such that $f(X)$ is totally bounded. For $U \in \mathscr{U}$, set $U' = \{f \in B(X,L)\colon f(X) \subset U\}$. Then $\mathscr{U}' = \{U'\colon U \in \mathscr{U}\}$ defines a locally convex, complete linear space topology for $B(X, L)$. Now let S be a topological semigroup with identity e. For $f \in B(S, L)$ and $a \in S$, let $f_a(s) = f(sa)$ and ${}_af(s) = f(as)$ for all $s \in S$. Finally, $CB(S, L)$ is the subset of continuous functions in $B(S, L)$.

The following, which is a generalization of a theorem of Veress about complex-valued functions, is basic to section 2. The key to the proof below was found in an unpublished proof of Veress' theorem given by Yood.

THEOREM 1. *Let $\{f_\gamma\colon \gamma \in G\} \subset B(X, L)$. Then $\{f_\gamma\}$ is totally bounded in the $\mathscr{U}'$ topology if and only if $\bigcup\{f_\gamma(X)\colon \gamma \in G\}$ is totally bounded and for each $U \in \mathscr{U}$ there is a partition $\{A_1, \ldots, A_n\}$ of X such that if $x, y \in A_t$, then $[f_\gamma(x) - f_\gamma(y)] \in U$ for all $\gamma \in G$.*

Proof. Necessity. Let $U \in \mathscr{U}$. Pick $V \in U$ so that $V + V - V - V \subset U$; then pick $W \in \mathscr{U}$ so that $W + W \subset V$. There are $\gamma(1), \ldots, \gamma(r) \in G$, such that $\{f_\gamma\} \subset \bigcup\{f_{\gamma(i)} + W'\colon i = 1, \ldots, r\}$, and for each i there exist $p(i, 1), \ldots, p(i, s_i)$ such that $f_{\gamma(i)}(X) \subset \bigcup\{p(i, j) + W\colon j = 1, \ldots, s_i\}$. So

(1) The material in this paper comprises part of the author's Ph. D. dissertation, which was submitted to the Graduate School of the University of Oregon in 1966. The author wishes to express his gratitude to his adviser, Professor Bertram Yood, for his advice and encouragement during and after the preparation of the original work.

$$\bigcup\{f_\gamma(X)\colon \gamma\in G\}\subset\bigcup\{f_{\gamma(i)}(X)+W\colon i=1,\dots,r\}$$
$$\subset\bigcup\big\{\bigcup\{p(i,j)+W+W\colon j=1,\dots,s_i\}\colon i=1,\dots,r\big\}$$
$$\subset\bigcup\{p(i,j)+V\}.$$

Thus the set $\bigcup\{f_\gamma(X)\colon \gamma\in G\}$ is totally bounded.

Now for each pair (i,j) let $E(i,j)=f_{\gamma_i}^{-1}\big(p(i,j)+V\big)$. Consider all sets of the form $E(1,j_1)\cap\dots\cap E(r,j_r)$, where $j_i\in\{1,\dots,s_i\}$: make them disjoint and label them $A_1,\dots,A_n$; these partition X. Finally, let $x,y\in A_t$; for any $\gamma\in G$, find i such that $f_\gamma-f_{\gamma_i}\in W'\subset V'$ and observe, since x and y are in an $E(i,j)$ for some j, that

$$f_\gamma(x)-f_\gamma(y)=f_\gamma(x)-f_{\gamma_i}(x)+f_{\gamma_i}(x)-f_{\gamma_i}(y)+f_{\gamma_i}(y)-f_\gamma(y)$$
$$\in W+\big(p(i,j)+V\big)-\big(p(i,j)+V\big)-W\subset V+V-V-V\subset U.$$

Sufficiency. Let $U\in\mathscr{U}$. Pick $V\in\mathscr{U}$ so that $V-V\subset U$; then find a $W\in U$ such that $W+W\subset V$. By hypothesis,

(1) there exist $p(1),\dots,p(n)\in L$ such that, for all $\gamma\in G$, $f_\gamma(X)\subset\big(p(1)+ +W\big)\cup\dots\cup\big(p(n)+W\big)$, and

(2) we can decompose X into $A_1,\dots,A_m$ so that $x,y\in A_i$ implies that $[f_\gamma(x)-f_\gamma(y)]\in W$ for all $\gamma\in G$.

Now let $\gamma\in G$. For each $i\in\{1,\dots,m\}$, find an element $y_i\in A_i$. Using (1), pick $k_{\gamma,i}$ such that $f_\gamma(y_i)\in\big(p(k_{\gamma,i})+W\big)$. Then by (2), if $x\in A_i$, we have

$$f_\gamma(x)\in f_\gamma(y_i)+W\subset\big(p(k_{\gamma,i})+W+W\big)\subset\big(p(k_{\gamma,i})+V\big),$$

so that $f_\gamma(A_i)\subset\big(p(k_{\gamma,i})+V\big)$. By this process we can define, for each γ, an m-tuple $\pi(\gamma)=(k_{\gamma,1},\dots,k_{\gamma,m})$, where $f_\gamma(A_i)\subset\big(p(k_{\gamma,i})+V\big)$. Enumerate these m-tuples; say they are indexed by $\{1,\dots,s\}$ (where $s\leqslant n^m$). Next, for each $j\in\{1,\dots,s\}$, let γ_j be any $\gamma\in G$ for which $\pi(\gamma_j)$ is the j-th m-tuple in this enumeration.

Then for any $\gamma\in G$ pick γ_j so that $\pi(\gamma_j)=\pi(\gamma)$. For $x\in X$, x is in, say, A_i, and since γ and γ_j have the same m-tuple, both $f_\gamma(A_i)$ and $f_{\gamma_j}(A_i)$ are subsets of $p(k_{\gamma,i})+V$. Therefore

$$f_\gamma(x)-f_{\gamma_j}(x)\in\big(p(k_{\gamma,i})+V\big)-\big(p(k_{\gamma,i})+V\big)=V-V\subset U;$$

thus $f_\gamma-f_{\gamma_j}\in U'$, which implies that

$$\{f_\gamma\colon \gamma\in G\}\subset(f_{\gamma_1}+U')\cup\dots\cup(f_{\gamma_s}+U').$$

2. Two definitions and some relationships.

Definition 1 (von Neumann [13], de Leeuw and Glicksberg [2]). $AP1(S,L)$ is the subset of $CB(S,L)$ such that $f\in AP1(S,L)$ if and only if $\{f_a\colon a\in S\}$ is totally bounded in the $\mathscr{U}'$ topology.

Using Theorem 1, one can easily show that replacing $\{f_a: a \in S\}$ by $\{{}_af: a \in S\}$ or $\{{}_af_b: a, b \in S\}$ yields the same set of functions.

Definition 2 (Maak [9]). $AP2(S, L)$ is the subset of $CB(S, L)$ such that $f \in AP2(S, L)$ if and only if for each $U \in \mathscr{U}$ there exists a partition $P(f, U) = \{A_1, \ldots, A_n\}$ of S so that if $a_0xb_0, a_0yb_0 \in A_t$ for some a_0, b_0, then $[f(cxd) - f(cyd)] \in U$ for all $c, d \in S$.

It follows from Theorem 1 that $AP2(S, L) \subset AP1(S, L)$, and that, for a group G, $AP2(S, L) = AP1(S, L)$; in the latter case we write $AP(G, L)$ for either set and observe that $AP(G, L)$ has been intensely studied [13]. When L is the complex number field, we write just $AP1(S)$, $AP(G)$, etc., as appropriate.

Definition 3. For S as in Definition 0, S_0 will signify the semigroup S given the discrete topology.

We note that Definition 2 differs from Maak's definition of almost periodic functions only in the following two particulars:

(i) the domain of the functions considered here is given a topology, and

(ii) the range of the present functions is more general.

For the discrete semigroup S_0, difference (i) is irrelevant and the modifications to Maak's work connected with (ii) are routine (for an indication, see Iseki [7]). Thus we have.:

THEOREM 2 (Maak [9]). *The translation operators $\{T_a: a \in S\}$ on $AP2(S_0, L)$ given by $T_a(f) = f_a$ are one-to-one mappings onto $AP2(S_0, L)$ and so generate a group (with respect to composition) G^* of operators on $AP2(S_0, L)$. The mapping $a \to T_a$ is a homomorphism of S_0 into G^*. Moreover, there is a one-to-one linear mapping $f \to f^*$ of $AP2(S_0, L)$ onto $AP(G^*, L)$ given by*

$$f^*(T) = Tf(e) \quad \text{for all } T \in G^*,$$

where $f(x) = f^*(T_x)$ for all $x \in S_0$.

(Comment: there is a slip in the proof of Theorem 7, p. 53, [9], but it can be corrected by methods used in that paper.)

In addition to Theorem 2, all results of [9] which are appropriate will be taken for granted as they apply to $AP2(S_0, L)$:

Definition 4. Set $A^* = \{f^*: f \in AP2(S, L)\}$. (Note that $A^* \subset AP(G^*, L)$, and if S is discrete, then $A^* = AP(G^*, L)$.) Then let G be the group G^* given the weak topology induced by the functions in A^*.

With this definition, the homomorphism $a \to T_a$ from S into G defined in Theorem 2 is continuous. Further, it is clear that $AP(G, L) \subset A^*$. The interesting fact is that $AP(G, L) = A^*$. When this is established, we shall have a nice extension of Theorem 2 for the topological semigroup S.

In [9], Maak proves that whenever $f \epsilon AP2(S_0, K)$ (K = complex numbers), there exist functions $f(\cdot, y)$ on S_0 determined by the following properties:

(i) $f(\cdot, y) \epsilon AP2(S_0, K)$ for each $y \epsilon S_0$;

(ii) $f(x, e) = f(x)$ for every $x \epsilon S_0$;

(iii) $f(xa, ya) = f(x, y)$ for all $a, x, y \epsilon S_0$.

The proof runs along the following lines. As a result of a complicated combinatorial process it is found (Theorem 5, p. 43, [9]) that if $x, a \epsilon S_0$ and $\varepsilon > 0$, then there is an $a' \epsilon S_0$ such that

$$|f(cad) - f(cxa'd)| < \varepsilon \text{ for all } c, d \epsilon S_0.$$

Thus, if $y \epsilon S_0$, one can find, for any $\varepsilon > 0$, a $y' \epsilon S_0$ such that

$$|f(cd) - f(cyy'd)| < \varepsilon \quad \text{for all } c, d \epsilon S_0.$$

Using this y' the function $f_\varepsilon(\cdot, y)$ is defined by $f_\varepsilon(x, y) = f(xy')$. It is shown that as $\varepsilon \to 0$, the functions $f_\varepsilon(\cdot, y)$, so constructed, form a Cauchy net in the uniform norm and hence converge to a function $f(\cdot, y)$, which turns out to have the stated properties.

This function takes the place of the function $f(xy^{-1})$ which can be defined when S_0 is a group, in the sense that, if y is invertible, then $f(x, y) = f(xy^{-1}, yy^{-1}) = f(xy^{-1}, e) = f(xy^{-1})$.

Just as $f(x, y)$ takes the place of $f(xy^{-1})$ in [9], we define a function to take the place of $f(y^{-1}x)$.

Definition 5. If $f \epsilon AP2(S_0, L)$, then there exist functions $f'(\cdot, y)$ from S_0 to L uniquely determined by the following properties:

(i) $f'(\cdot, y) \epsilon AP2(S_0, L)$ for each $y \epsilon S_0$;

(ii) $f'(x, e) = f(x)$ for every $x \epsilon S_0$;

(iii) $f'(ax, ay) = f'(x, y)$ for all $a, x, y \epsilon S_0$.

(The existence of these functions is demonstrated in a manner completely analogous to the methods of [9].)

PROPOSITION 1. *If $f \epsilon AP2(S_0, L)$ and $a \epsilon S_0$, then $f'(a, \cdot) \epsilon AP2(S_0, L)$.*

Proof. By Theorem 2, $f^* \epsilon AP(G^*, L)$; so, by the theory of almost periodic functions on groups, $f^*(T^{-1}T_a) \epsilon AP(G^*, L)$ (as a function of T), so if we set $g(a, y) = f^*(T_y^{-1}T_a)$, Theorem 2 shows that $g(a, \cdot) \epsilon AP2(S_0, L)$. On the other hand, for fixed y, $f^*(T_y^{-1}T) \epsilon AP(G^*, L)$, so again by Theorem 2, (i) $g(\cdot, y) \epsilon AP2(S_0, L)$; but (ii) and (iii) of Definition 5 are immediate for $g(\cdot, y)$, so $g(x, y) \equiv f'(x, y)$.

Now we can proceed to consideration of the topological semigroup S.

LEMMA 1. *If $f \epsilon AP2(S, L)$, then $\{{}_af_b : a, b \epsilon S\}$ is an equicontinuous family of functions.*

Proof. With the help of Theorem 1, this is routine.

COROLLARY. *If* $f \in AP2(S, L)$ *and* $a, b \in S$, *then* ${}_af_b \in AP2(S, L)$.

LEMMA 2. *If* $f \in AP2(S, L)$ *and* $a, b \in S$, *then* $f(\cdot, b)$, $f(a, \cdot)$, $f'(\cdot, b)$ *and* $f'(a, \cdot)$ *are all members of* $AP2(S, L)$.

Proof. Because of results in [9] and Proposition 1, it suffices to show these functions are continuous.

(a) In [9], the function $f(\cdot, b)$ is constructed as the uniform limit of functions of the form $f_{b'}$, each of which is continuous by Lemma 1.

(b) To see that $f(a, \cdot) \in AP2(S, L)$ let $y_0 \in S$ and $U \in \mathscr{U}$. Pick $V \in \mathscr{U}$ such that $V + V - V + V + V - V - V \subset U$. By Lemma 1 there is a neighborhood N of y_0 for which $y \in N$ implies

$$[f(cyd) - f(cy_0d)] \in V \quad \text{for all } c, d \in S.$$

By Theorem 5, p. 43, [9], there exists an $a' \in S$ such that

$$[f(cad) - f(ca'y_0d)] \in V \quad \text{for all } c, d \in S.$$

By definition [9] of $f(a, y)$, there exists a $y' \in S$ such that

$$[f(cd) - f(cyy'd)] \in V \quad \text{and} \quad [f(a, y) - f(ay')] \in V$$

for all $c, d \in S$; similarly, there is a $y_0' \in S$ with the property that

$$[f(cd) - f(cy_0y_0'd)] \in V \quad \text{and} \quad [f(a, y_0) - f(ay_0')] \in V$$

for all $c, d \in S$. Thus if $y \in N$,

$$\begin{aligned}
&[f(a, y_0) - f(a, y)] \\
&\quad = [f(a, y_0) - f(ay_0')] + [f(ay_0') - f(ay')] + [f(ay') - f(a, y)] \\
&\quad \in V + [f(ay_0') - f(a'y_0y_0')] + [f(a'y_0y_0') - f(a')] + [f(a') - f(a'yy')] + \\
&\qquad + [f(a'yy') - f(a'y_0y')] + [f(a'y_0y') - f(ay')] - V \\
&\quad \subset V + V - V + V + V - V - V \subset U.
\end{aligned}$$

(c) The proofs for $f'(a, \cdot)$ and $f'(\cdot, b)$ are essentially the same.

LEMMA 3. *If* $f^* \in A^*$ (*i.e., if* $f \in AP2(S, L)$) *and* $R \in G^*$, *then the functions* f^*_R, ${}_Rf^*$ *and* H, *where* $H(T) = f^*(T^{-1})$, *are all members of* A^*.

Proof. (a) For $T \in G^*$, $f^*_R(T) = f^*(TR) = TRf(e) = (Rf)^*(T)$.

(b) Suppose first that $R = T_aT_b^{-1}$, $a, b \in S$. By Theorem 2, it is enough to see that (as a function of x) ${}_Rf^*(T_x) \in AP2(S, L)$. To this end, consider the function ${}_{T_aT_y^{-1}}f^*(T_x)$ defined for all $x, y \in S$.

(i) For fixed y it is clear that ${}_{T_aT_y^{-1}}f^* \in AP(G^*, L)$, so by Theorem 2, ${}_{T_aT_y^{-1}}f^*(T_x) \in AP2(S_0, L)$ (as a function of x).

(ii) ${}_{T_aT_e^{-1}}f^*(T_x) = f^*(T_aT_x) = T_aT_xf(e) = f(ax) = {}_af(x)$.

(iii) ${}_{T_aT_{cy}^{-1}}f^*(T_{cx}) = f^*(T_aT_y^{-1}T_c^{-1}T_cT_x) = f^*(T_aT_y^{-1}T_x) = T_aT_y^{-1}f^*(T_x)$.

Thus by definition 5, ${}_{T_aT_y^{-1}}f^*(T_x) = ({}_af)'(x, y)$, and so by the corollary to Lemma 1 and Lemma 2, ${}_Rf^*(T_x) = {}_{T_aT_b^{-1}}f^*(T_x) = ({}_af)'(x, b) \in AP2(S, L)$ (as a function of x).

Now suppose the Theorem has been proved for all $T \in G^*$ of the form $T_k = T_{a_k}T_{b_k}^{-1} \dots T_{a_1}T_{b_1}^{-1}$, and consider an element of the form T_{k+1}; we can write $T_{k+1} = T_kT_aT_b^{-1}$ for some $T_k \in G^*$ and $a, b \in S$. Then

$$ {}_{T_{k+1}}f^*(T) = f^*(T_kT_aT_b^{-1}T) = {}_{T_k}f^*(T_aT_b^{-1}T) = {}_{T_aT_b^{-1}}({}_{T_k}f^*)(T); $$

and now the induction hypothesis and the first part of the proof apply.

(c) It is enough to show that $h \in AP2(S, L)$, where $h(x) = H(T_x)$;

$$ h(x) = f^*(T_x^{-1}) = T_x^{-1}f(e) = f(e, x). $$

COROLLARY. *G is a topological group.*

LEMMA 4. $A^* = AP(G, L)$.

Proof. Let $F \in AP(G, L)$ and let $T\colon x \to T_x$. By Theorem 2, $f = F \circ T \in AP2(S_0, L)$, so as the composition of continuous functions, $f \in AP2(S, L)$. But again by Theorem 2, $f^* = F$.

A summary of the results so far is

THEOREM 4. *Let S, S_0, L and $AP2(S, L)$ be as in Definitions 0, 2 and 3. Then there is a topological group G of one-to-one transformations of $AP2(S_0, L)$ onto itself generated by the right translation mappings $\{T_a : a \in S\}$ such that the mapping $a \to T_a$ is a continuous homomorphism of S into G. Moreover, there is a one-to-one linear mapping $f \to f^*$ of $AP2(S, L)$ onto $AP(G, L)$ given by*

$$ f^*(T) = Tf(e) \quad \textit{for all } T \in G, $$

where $f(x) = f^*(T_x)$ for all $x \in S$.

As noted above, for a group G, $AP1(G, L) = AP2(G, L)$. For semigroups this is not the case; in [2] it is shown that all semicharacters (bounded multiplicative functionals) belong to $AP1(S)$, and that all semigroups examined therein admit semicharacters taking on values of modulus less than 1.

PROPOSITION 2. *If S admits a semicharacter $\gamma \neq 0$ such that $|\gamma(x)| < 1$ for some x, then $AP1(S) \neq AP2(S)$.*

Proof. Since $\gamma \neq 0$, $\gamma(e) = 1$. Suppose $\gamma \in AP2(S)$. Find n such that $|\gamma(x^n)| = |\gamma(x)|^n < 1/2$ and set $y = x^n$. Now let $\varepsilon = 1/2$ if $\gamma(y) = 0$ and $\varepsilon = |\gamma(y)|$ otherwise. As in [9], there exists $y' \in S$ such that

$$ \varepsilon > |\gamma(e) - \gamma(yy')| = \big|1 - |\gamma(y)| \cdot |\gamma(y')|\big|, $$

which is impossible if $\gamma(y) = 0$; so $\varepsilon = |\gamma(y)|$ and we have

$$1 > \left| |\gamma(y)|^{-1} - |\gamma(y')| \right|;$$

but $|\gamma(y)|^{-1} > 2$, so $|\gamma(y')| > 1$, which is impossible, since semicharacters are bounded.

In a more restricted setting it is possible to make the relation between $AP1(S)$ and $AP2(S)$ much more definite. Let us say that Restriction 1 is satisfied by a semigroup S in case it is commutative and contains an infinite set X with the property that if $a, c \in S$, then $\{x \in X: xc \notin Sa\}$ is finite. (E.g., cones in E_n, finitely generated commutative semigroups.)

LEMMA 5. *Let S satisfy Restriction* 1 *and let $f \in AP1(S)$. Then the closure of $\{f_x: x \in X\}$ meets $AP2(S)$.*

Proof. (Note that here the $\mathscr{U}'$ topology is that provided by the sup norm $\|f\| = \sup\{|f(s)|: s \in S\}$.) Since $\{f_x: x \in S\}$ is totally bounded, it contains a Cauchy sequence $\{f_{x_n}\}$ which converges to a function $g \in CB(S)$. We show that $g \in AP2(S)$.

Let $\varepsilon > 0$ be given. By Theorem 1 there is a partition $\{A_1, \ldots, A_n\}$ of S such that if $z, w \in A_t$, then $|f_s(z) - f_s(w)| < \varepsilon/3$ for all $s \in S$. Now suppose $ax, ay \in A_t$, and let $c \in S$. Find n such that $x_n c = pa$ for some $p \in S$ and $\|g - f_{x_n}\| < \varepsilon/3$. Then

$$\begin{aligned} |g(cx) - g(cy)| &\leqslant |g(cx) - f_{x_n}(cx)| + |f_{x_n}(cx) - f_{x_n}(cy)| + |f_{x_n}(cy) - g(cy)| \\ &< \varepsilon/3 + |f_p(ax) - f_p(ay)| + \varepsilon/3 < \varepsilon; \end{aligned}$$

i.e., $\{A_1, \ldots, A_n\}$ is a partition $P(f, \varepsilon)$ for Definition 2.

Theorem 4, p. 49, [9], states that if $f \in AP2(S_0)$, $\varepsilon > 0$, and $T \in G^*$ then there exists an element $a \in S_0$ such that

$$|Tf(x) - T_a f(x)| \leqslant \varepsilon \quad \text{for all } x \in S_0.$$

As an easy consequence of this, if $f \in AP2(S)$, then $\|f^*\| = \|f\|$ and $f \geqslant 0$ implies $f^* \geqslant 0$. These facts will be useful later.

THEOREM 5. *Let S satisfy Restriction* 1 *and $f \in AP1(S)$. Then $f = h + k$ uniquely, where $h \in AP2(S)$ and* $\inf\{\|k_x\|: x \in S\} = 0$.

Proof. Lemma 5 provides a function $g \in AP2(S)$ and a sequence $\{x_n\} \subset S$ such that $\|f_{x_n} - g\| \to 0$. Since $(T_x^{-1} g)^* = (g^*)_{T_x}$, the sequence $\{(T_{x_n}^{-1} g)^*\}$ must contain a convergent subsequence $\{(T_{x_j}^{-1} g)^*\}$. Then $\{T_{x_j}^{-1} g\}$ also converges, say to $h \in AP2(S)$. Set $k = f - h$; then

$$\begin{aligned} \|k_{x_j}\| &= \|(f - T_{x_j}^{-1} g + T_{x_j}^{-1} g - h)_{x_j}\| \\ &\leqslant \|f_{x_j} - g\| + \|T_{x_j}^{-1} g - h\| \to 0. \end{aligned}$$

Uniqueness follows at once from Theorem 3, p. 42, [9].

Because of Theorem 5, we can write $AP1(S) = AP2(S) \oplus N(S)$, where $N(S) = \{f \in AP1(S) \colon \inf \|f_x\| = 0\}$. In case S is the non-negative reals, $N(S) = C_0(S)$ and this becomes a result of Fréchet [3]. (In case $S = \{(x, y) \colon x, y \geqslant 0\}$, the function $f(x, y) = e^{-x}$ shows that $N(S) \neq C_0(S)$.)

Each commutative semigroup S admits an invariant mean for $CB(S)$; i.e., a linear functional M such that ([6], p. 231)

(1) $M(1) = 1$,

(2) $M(f_x) = M(f)$ for all $f \in CB(S)$ and all $x \in S$,

(3) if $f \geqslant 0$, then $M(f) \geqslant 0$.

LEMMA 6. *Let S be any commutative topological semigroup and let M be as above. If M^* is defined on $AP(G)$ by $M^*(f^*) = M(f)$, where $f \to f^*$ and G is as in Theorem 4, then M^* is the usual mean value for $AP(G)$.*

Proof. Conditions (1)-(3) characterize the usual mean among linear functionals on $AP(G)$; it is clear that M^* satisfies (1) and (3).

Ad (2). Let $f^* \in AP(G)$ and $T = T_a T_b^{-1} \in G$. Then if $g \to f_T^*$, a simple calculation shows that $g(x) = f(xa, b)$. Now $f(xa, b) = \lim_{\varepsilon \to 0} f(xab_\varepsilon)$, and the convergence is uniform ([9], p. 45), so

$$M^*(f_T^*) = M(g) = M(\lim f_{ab_\varepsilon}) = \lim M(f_{ab_\varepsilon}) = M(f) = M^*(f^*).$$

THEOREM 6. *With S and X as in Restriction* 1, *and $N(S)$ as above,* $N(S) = \{f \in AP1(S) \colon M(|f|) = 0\}$.

Proof. It $f \in N(S)$, then $M(|f|) = M(|f_x|) \leqslant \|f_x\| \to 0$ for appropriate x's. Now let $f \in AP1(S)$, $M(|f|) = 0$. By Theorem 5, $f = h + k$, where $h \in AP2(S)$ and $k \in N(S)$; then $M(|h|) \leqslant M(|f|) + M(|k|)$. Therefore $M^*(|h|^*) = M(|h|) = 0$; but $|h| \geqslant 0$ implies $|h|^* \geqslant 0$, so by Theorem 7, p. 452, [13], $|h|^* = 0$. Thus $|h| = 0$, so $h = 0$ and $f = k$.

Remark. This is another result of Frechet when S is the non-negative reals.

3. Algebras of generalized almost periodic functions. In the uniform norm, with convolution multiplication, $AP(G)$ is a Banach algebra whose structure is known in detail ([10], [15]). For the case $G = R$, the real line, a number of metrics have been defined by Stepanoff, Weyl, and Besicovitch ([1], Ch. II) under which $AP(G)$ is a normed linear space (not necessarily complete). Analogues of these metrics can be defined on $AP(G, A)$, where G is a locally compact group and A is a Banach algebra; in this section we take a look at the Banach algebras formed by completing $AP(G, A)$ in these new metrics.

Let G and A be as just above. There is a compact group G_B, the Bohr compactification of G, and a continuous homomorphism $\pi \colon G \to G_B$ with $\pi(G)$ dense in G_B ([8], p. 168). Grove [4] shows that $AP(G, A)$ is isometrically isomorphic (as a Banach algebra) with $C(G_B, A)$ under a correspondence $f \to f^{\wedge}$, where it is implicit in the isomorphism that $f^{\wedge}(\pi(x)) = f(x)$ for all $x \in G$. It is also noted therein that for $f \in AP(G, A)$,

$$M(f) = \int_{G_B} f^{\wedge}(\gamma)\, d\gamma,$$

where M is the mean value [14] for $AP(G, A)$ (we shall use Greek letters for elements of G_B). In light of this we have the convolution

$$f * g(s) = M_t\big(f(st^{-1})g(t)\big) = \int_{G_B} f^{\wedge}\big(\pi(s)\gamma^{-1}\big) g^{\wedge}(\gamma)\, d\gamma$$

whenever $f, g \in AP(G, A) \cup AP(G)$ and the product $f(st^{-1})g(t)$ makes sense. It is shown just as for $AP(G)$ that $f * g \in AP(G, A)$.

Definition 6. Let S be any compact subset of G (not necessarily a semigroup) with $\lambda(S) > 0$ (Haar measure), let $\mathscr{W}$ be any sequence of compact subsets of G each having positive measure, and let $1 \leqslant p < \infty$. Then for $f \in AP(G, A)$ we define

$$\|f\|_{S^p} = \sup\Big\{[1/\lambda(S) \int_{aSb} \|f(s)\|_A^p\, ds]^{1/p} : a, b \in G\Big\},$$

$$\|f\|_{W^p} = \limsup \{\|f\|_{S^p} : S \in \mathscr{W}\},$$

$$\|f\|_{B^p} = \big(M_t[\|f(t)\|_A^p]\big)^{1/p} = \Big(\int_{G_B} \|f(\gamma)\|_A^p\, d\gamma\Big)^{1/p}$$

(where $\| \ \|_A$ denotes the norm in A).

In case $G = R$ and A is the complex numbers, these become, with suitable choices for the compact sets, the metrics of Stepanoff, Weyl, and Besicovitch, respectively. Minkowski's inequality shows that they are norms on $AP(G, A)$ in any case. Also, since $\|f * g\|_\infty \leqslant \|f\|_\infty \|g\|_\infty$, the usual proof shows that $f * g \in AP(G, A)$ whenever $f \in AP(G) \cup AP(G, A)$ and $g \in AP(G, A)$.

THEOREM 7. *If $\| \ \|_N$ denotes any of the norms of Definition 6 or the sup norm, then*

(i) $$\|f\|_{B^1} \leqslant \|f\|_N \leqslant \|f\|_\infty,$$

(ii) $$\|f * g\|_N \leqslant \max(\|f\|_{B^1} \|g\|_N, \|f\|_N \|g\|_{B^1}).$$

Proof. (ii) First consider $f \in AP(G)$ and $\theta \in L_\infty(G)$, and let $a \in G_B$ and $\varepsilon > 0$. Since $f^{\wedge}$ if left uniformly continuous on G_B, there exists a neighborhood γ of $\pi(e)$ such that if $\delta^{-1} a \in \gamma$, then $|f^{\wedge}(a) - f^{\wedge}(\delta)| < \varepsilon / \|\theta\|_\infty \lambda(S)$, and because $\pi(G)$ is dense in G_B, there is a $c \in G$ such that $\pi(c) \in a\gamma$, so for all $s \in G$,

$$\big|f^{\wedge}\big(\pi(s)a\big) - f^{\wedge}\big(\pi(s)\pi(c)\big)\big| < \varepsilon / \|\theta\|_\infty \lambda(S).$$

Thus

$$\int_S \left|f^\wedge\big(\pi(s)a\big)\theta(s)\right|ds$$

$$\leqslant \int_S \left|f^\wedge\big(\pi(s)\pi(c)\big)\theta(s)\right|ds + \int_S \left|f^\wedge\big(\pi(s)\pi(c)\big)-f^\wedge\big(\pi(s)a\big)\right|\,|\theta(s)|\,ds$$

$$< \int_S |f(sc)\theta(s)|\,ds + \varepsilon.$$

Now let $1 \leqslant p < \infty$; for $a, b \in S$ and $\varepsilon > 0$, take $\theta \in CB(aSb)$ with $\|\theta\|_q \leqslant 1$ $(q = p/(1-p))$. Then

$$\Big|\int_{aSb} \|f*g(s)\|_A\,\theta(s)\,ds\Big|$$

$$= \Big|\int_{aSb}\Big\|\int_{G_B} f^\wedge\big(\pi(s)\gamma^{-1}\big)g^\wedge(\gamma)\,d\gamma\Big\|_A\theta(s)\,ds\Big|$$

$$\leqslant \int_{G_B}\|g^\wedge(\gamma)\|_A \int_{aSb}\big\|f^\wedge\big(\pi(s)\gamma^{-1}\big)\big\|_A\,|\theta(s)|\,ds\,d\gamma$$

$$\leqslant \int_{G_B}\|g^\wedge(\gamma)\|_A\Big(\int_{aSb}\|f(sc)\|_A\,|\theta(s)|\,ds+\varepsilon\Big)d\gamma$$

$$\leqslant \int_{G_B}\|g^\wedge(\gamma)\|_A\Big(\Big(\int_{aSb}\|f(sc)\|_A^p\,ds\Big)^{1/p}+\varepsilon\Big)d\gamma$$

$$\leqslant \|g\|_{B^1}\big(\lambda(S)^{1/p}\|f\|_{S^p}+\varepsilon\big).$$

But ε was arbitrary, so

$$\Big|\int_{aSb}\|f*g(s)\|_A\,\theta(s)\,ds\Big| \leqslant \|g\|_{B^1}\|f\|_{S^p}\lambda(S)^{1/p}.$$

Now taking the sup over all such $\theta \in CB(aSb)$, dividing by $\lambda(S)^{1/p}$, then taking the sup over all $a, b \in G$ yields

$$\|f*g\|_{S^p} \leqslant \|f\|_{S^p}\|g\|_{B^1}.$$

All other parts of (ii) are proved similarly, follow easily, or are adaptations of standard proofs for complex-valued functions.

(i) Consider the functions $1(s) \equiv 1$ and $\|f(\cdot)\|_A$, both in $AP(G)$. We have

$$1*\|f(\cdot)\|_A(s) = M\big(\|f(\cdot)\|_A\big) = \|f\|_{B^1},$$

so that

$$\|f\|_{B^1} = \big\|1*\|f(\cdot)\|_A\big\|_{B^1} \leqslant \|1\|_{B^1}\big\|\|f(\cdot)\|_A\big\|_{S^1} = \|f\|_{S^1}.$$

Hölder's inequality is the only other thing needed to complete the proof.

The first consequence of this theorem is that $\| \ \|_N$ satisfies the continuity requirement for normed algebra multiplication.

Definition 7. Let $\| \ \|_N$ be any of the norms of Definition 6. By $AN(G, A)$ we mean the Banach algebra formed by completing $AP(G, A)$ in $\| \ \|_N$.

COROLLARY 1. *If* $f \in AP(G) \cup AP(G, A)$ *and* $g \in AN(G, A)$, *then* $f*g \in AP(G, A)$.

Proof. Say $g_n \xrightarrow{N} g$, each $g_n \in AP(G, A)$. Then

$$\|f*g - f*g_n\|_\infty = \|f*(g-g_n)\|_\infty \leqslant \|f\|_\infty \|g-g_n\|_N \to 0,$$

and $AP(G, A)$ is uniformly closed.

COROLLARY 2. *If* $f \in AN(G, A)$ *and* $\varepsilon > 0$, *then there exists* $g \in AP(G)$ *such that* $\|g*f - f\|_N < \varepsilon$.

Proof. Pick $f_0 \in AP(G, A)$ such that $\|f - f_0\|_N < \varepsilon/3$. Then, as shown in [13], we can find $g \in AP(G)$ with the property that $\|g\|_{B^1} = 1$ and $\|g*f_0 - f_0\|_\infty < \varepsilon/3$. It follows that

$$\begin{aligned}\|g*f - f\|_N &\leqslant \|g*f - g*f_0\|_N + \|g*f_0 - f_0\|_N + \|f_0 - f\|_N \\ &\leqslant \|g\|_{B^1} \|f - f_0\|_N + \|g*f_0 - f_0\|_\infty + \|f - f_0\|_N \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

These corollaries, along with the isometry between $AP(G, A)$ and $CB(G_B, A)$, permit us to relate many questions about $AN(G, A)$ to the (more familiar?) algebra of continuous functions. Some results along these lines, due to Spicer [11], are of the following type:

(i) $AN(G, A)$ is (Jacobsen) semi-simple if and only if $AP(G, A)$ is semi-simple,

(ii) if $AN(G, A)$ is semi-simple, then A is semi-simple,

(iii) if A is a semi-simple annihilator algebra, then $AN(G, A)$ is the direct topological sum of its minimal cosed ideals, which are precisely the N-norm closures of the minimal closed ideals of $AP(G, A)$.

An unresolved question in [11] is answered in the following:

THEOREM 8. *Suppose G is a compact group and A is a semi-simple Banach algebra. Then $C(G, A)$ (convolution multiplication) is semisimple.*

Proof. *Step one.* Let $a \to Ta$ be an irreducible representation of A by bounded operators on the Banach space X and $x \to U_x$ be a continuous irreducible unitary representation of G on n-dimensional complex space K^n. Note that U_x can be thought of as an $(n \times n)$-matrix of complex numbers such that, for the orthonormal basis $\xi_1, \ldots, \xi_n$ for K^n, the inner product $\langle U_x \xi_j, \xi_i \rangle$ is the (i, j)-th entry of U_x. Now define $TU: C(G, A) \to$

$B(X^n)$ as follows: $TU(f)$ "is" an $(n \times n)$-matrix whose (i, j)-th entry is

$$[TU(f)]_{ij} = T \int_G \overline{\langle U_x \xi_j, \xi_i \rangle} f(x)\, dx.$$

It is clear that TU is a linear mapping. The verification that TU is multiplicative is a routine application of Fubini's Theorem to a problem in matrix multiplication.

Now consider two column vectors, $(x_1, \ldots, x_n)^t \neq 0$ and $(y_1, \ldots, y_n)^t$ in X^n. Pick $m \in \{1, \ldots, n\}$ such that $x_m \neq 0$. Since T is irreducible, for each $k \in \{1, \ldots, n\}$ there is an $a_k \in A$ for which $Ta_k(x_m) = y_k$. Define $f \in C(G, A)$ by

$$f(x) = n \sum_{k=1}^{n} \langle U_x \xi_k, \xi_m \rangle a_k.$$

Then

$$\begin{aligned} [TU(f)]_{ij} &= T \int_G \overline{\langle U_x \xi_j, \xi_i \rangle}\, n \sum_{k=1}^{n} \langle U_x \xi_k, \xi_m \rangle a_k\, dx \\ &= \begin{cases} Ta_k & \text{if } j = k \text{ and } i = m, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

That is, $TU(f)$ has $Ta_1, \ldots, Ta_n$ in its m-th column and zeros elsewhere, so

$$TU(f)\big((x_1, \ldots, x_n)^t\big) = \big(Ta_1(x_m), \ldots, Ta_n(x_m)\big)^t = (y_1, \ldots, y_n)^t.$$

Thus TU is irreducible.

Step two. Let f be in the radical of $C(G, A)$. Then $TU(f) = 0$ for all T and U as in step one; i.e.,

$$T \int_G \overline{\langle U_x \xi_j, \xi_i \rangle} f(x)\, dx = 0$$

for all T, U, i, and j. But since A is semi-simple, this implies that

$$\int_G \overline{\langle U_x \xi_j, \xi_i \rangle} f(x)\, dx = 0$$

for all U, i, j. Hence, if L is a bounded linear functional on A,

$$\int_G \overline{\langle U_x \xi_j, \xi_i \rangle} Lf(x)\, dx = L \int_G \overline{\langle U_x \xi_j, \xi_i \rangle} f(x)\, dx = 0 \quad \text{for all } U, i, j.$$

Now, since the functions $\overline{\langle U_x \xi_j, \xi_i \rangle}$ are linearly dense in $C(G)$, the Riesz representation theorem shows that $Lf(x) = 0$ for all $x \in G$. But L was an arbitrary linear functional, so by the Hahn-Banach theorem, $f(x) = 0$ for all $x \in G$. Thus $C(G, A)$ is semi-simple.

COROLLARY. *If G is a topological group and A is a semi-simple Banach algebra, then $AN(G, A)$ is also a semi-simple Banach algebra.*

Remark 1. The method of proof for Theorem 8 was suggested in an article by Hausner [5]

Remark 2. While this manuscript was in preparation, Spicer [12] communicated that he had independently solved the problem attacked in Theorem 8. His approach, which is quite different, is to consider $B^1(G, A)$ as the tensor product of $L^1(G)$ with A, then work on the fine structure by largely algebraic means.

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