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ITERATED PRODUCTS OF IDEALS OF BOREL SETS

BY

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The well-known Fubini Theorem ([3]) describes a relation between the Lebesgue measure on the real line and on the plane. As a consequence a subset $A \subseteq I \times I$ is of positive measure in the unit square $I \times I$ if and only if A has sections $xA = \{y \in I;\ (x, y) \in A\}$ of positive measure (in the unit interval I) for all x belonging to a subset of positive measure (in I).

Thus, the ideal of all Borel sets of zero measure in $I \times I$ is equal to the product $L \times L$ of the ideal L of all Borel sets of zero measure in I, by itself. Similarly, the ideal of all Borel sets of the first category in $I \times I$ equals to the product $K \times K$ of the ideal K of all Borel sets of the first category in I: a Borel subset $A \subseteq I \times I$ is not of the first category in $I \times I$ exactly when $\{x \in I;\ xA \notin K\} \notin K$, i.e. when $A \notin K \times K$ (see e.g. [5]).

These facts imply that the products $L \times L$, $K \times K$ are ω-complete ω-saturated ideals in the field of all Borel sets in $I \times I$. It is not a very interesting result because the standard homeomorphism of $I \times I$ to I transformes $L \times L$ to L and $K \times K$ to K. But we can make mixed products of ideals, $L \times K$ or $K \times L$, in $I \times I$ and get (by the mentioned homeomorphism) ω-complete ω-saturated ideals of Borel sets in I, other than L and K. Further ideals can be formed by products of more factors, such as $L \times K \times L$, or $K \times L \times K$. However, the construction is not straightforward and requires use of special properties of ideals L, K.

The description of the iterated product of ideals L, K is, in a more general form, given in the second section of this paper. The necessary properties of L, K are examined and generalized in the first section. The third section is devoted to applications of iterated products for construction of various complete Boolean products of Cantor algebra $\mathscr{C}$ and random algebra $\mathscr{R}$. It is shown that, for algebras $\mathscr{C}$, $\mathscr{R}$, there are infinitely many incomparable complete Boolean products which, moreover, are locally disjoint. (The local disjointness of factors in complete Boolean product is equivalent with the disjointness of the corresponding cogeneric Boolean-valued models of the set-theory, see [1], [2]). The constructed products give solution of the problem, whether the minimal $(\mathfrak{m}, 0)$-product is necessarily the least one, [6]. At the end of the paper, an open problem of isomorphism for ideals, such as $\boldsymbol{L}\times\boldsymbol{K}$, $\boldsymbol{K}\times\boldsymbol{L}$ is formulated.

The notions and denotations are mostly standard, and follow [6] for Boolean notions (with the exception of denotation $\wedge$, $\vee$, 0, 1 for the Boolean operation and the bound elements). For set-theoretical denotation, an ordinal is considered as the set of all lesser ordinals, e.g. $\{0, 1\} = 2$, $\{0, 1, 2, \dots\} = \omega$, the set of all functions from A to B is denoted by ${}^AB$. For ordinals α, β we denote by ${}^{<\alpha}\beta$ the sum $\bigcup\{{}^{\gamma}\beta;\ \gamma < \alpha\}$. The domain of a function f is denoted $\mathscr{D}(f)$, the range $W(f)$. The set of all finite subsets of X is denoted by $P^{<\omega}(X)$. If X is a topological space, then $\mathscr{B}(X)$ denotes the field of all Borel subsets of X. The random algebra is the complete Boolean algebra $\mathscr{R} = \mathscr{B}(\boldsymbol{I})/\boldsymbol{L}$. Cantor algebra is $\mathscr{C} = \mathscr{B}(\boldsymbol{I})/\boldsymbol{K}$.

1. Quasimaximal systems of quasi-ideals. If X is the Cartesian product of sets X_0, X_1 and if I_0, I_1, I are ideals in fields $\mathscr{B}_0$, $\mathscr{B}_1$, $\mathscr{B}$ of subsets of spaces X_0, X_1, X respectively, then, by the standard definition of products for ideals, $I = I_0 \times I_1$ holds, if for any $A \in \mathscr{B}$,

$$A \notin I \equiv \{x_0 \in X_0;\ \{x_1 \in X_1;\ (x_0, x_1) \in A\} \notin I_1\} \notin I_0$$

is fulfilled.

This definition is meaningful when the following two conditions are satisfied:

(*) for any $A \in \mathscr{B}$, $x_0 \in X_0$ the set $x_0 A = \{x_1 \in X_1;\ (x_0, x_1) \in A\}$ belongs to $\mathscr{B}_1$,

(**) for any $A \in \mathscr{B}$, the set $AI_1 = \{x_0 \in X_0;\ x_0 A \notin I_1\}$ belongs to $\mathscr{B}_0$.

Let us suppose that X_0, X_1, X are topological spaces (the topology in X being the product topology) and $\mathscr{B}_0$, $\mathscr{B}_1$, $\mathscr{B}$ are the fields of all Borel sets in X_0, X_1, X, respectively. The condition (*) is satisfied under this assumption (see e.g. [5]).

To verify the condition (**) we proceed by induction through the hierarchy of Borel sets in $\mathscr{B}$. Let a be a countable subset of $\mathscr{B}$, such that for any $A \in a$ or $-A \in a$, (**) is fulfilled. It is easy to verify that, for any $x_0 \in X_0$,

$$x_0\left(\bigcup a\right) = \bigcup\{x_0 A;\ A \in a\}, \qquad x_0\left(\bigcap a\right) = \bigcap\{x_0 A;\ A \in a\},$$

and, therefore,

$$\left(\bigcup a\right) I_1 = \{x_0 \in X_0;\ \bigcup\{x_0 A;\ A \in a\} \notin I_1\},$$
$$\left(\bigcap a\right) I_1 = \{x_0 \in X_0;\ \bigcap\{x_0 A;\ A \in a\} \notin I_1\}.$$

If the ideal I_1 is ω-complete, we get $\left(\bigcup a\right) I_1 = \bigcup\{AI_1;\ A \in a\}$. Thus (**) is fulfilled for $\bigcup a$, in view of the induction assumption. But, the ω-completeness of I_1 does not suffice to prove $\left(\bigcap a\right) I_1 = \bigcap\{AI_1;\ A \in a\}$ and

(**) for $\bigcap a$. If the ideal I_1 is, moreover, maximal, we may "go to complements" and we get $(\bigcap a)I_1 = \bigcap\{-(-A)I_1;\ A\in a\}$ which, by induction assumption, gives (**) for $\bigcap a$.

The assumption of the maximality is too strong for our purpose, as we are preparing to multiply the ideals $\boldsymbol{L}$ and $\boldsymbol{K}$. We shall use another way to "come to complements". The idea is the following.

It is known that a Borel subset A of the real unit interval $\boldsymbol{I}$ is not of the first category in $\boldsymbol{I}$, exactly when there exists an interval $J \subseteq \boldsymbol{I}$ such that $-A$ is of the first category in J. We may suppose that the endpoints of J are rational. If we denote by $\boldsymbol{K}^J$ the ideal of all Borel subsets $A \subseteq \boldsymbol{I}$ such that $J \cap A \in \boldsymbol{K}$ and if $\boldsymbol{J}$ denotes the set of all subintervals of $\boldsymbol{I}$ with the rational endpoints, then we have

$$A \notin \boldsymbol{K} \equiv (\exists J \in \boldsymbol{J}) - A \in \boldsymbol{K}^J.$$

Moreover, any ideal $\boldsymbol{K}^J$ can be approximated in this dual way by ideals $\boldsymbol{K}^{J'}$, $J' \in \boldsymbol{J}$, $J' \subseteq J$. A similar pass to complements that resembles maximality is possible for the ideal $\boldsymbol{L}$. The sets $\boldsymbol{L}^r$ of all Borel subsets of measure at most r, for r rational, enable dual approximation

$$A \notin \boldsymbol{L}^r \equiv (\exists s \in Q,\ s < 1-r) - A \in \boldsymbol{L}^s.$$

However, $\boldsymbol{L}^r$ are not ideals (except for $\boldsymbol{L}^0 = \boldsymbol{L}$).

To make use of the above observations, we define a more general concept, called quasi-ideal. We shall multiply quasi-ideals not as single factors but as systems admitting dual approximation, the so-called quasi-maximal systems. Further, the notion of the ω-saturatedness will be modified, as to be preserved in the iterated product.

If $\mathscr{B}$ is a ω-complete field of subsets of a space X, then a subset $\mathscr{U} \subseteq \mathscr{B}$ is called a *ω-complete quasi-ideal* in $\mathscr{B}$, if

(i) $\mathscr{U}$ is *convex downward*, i.e. for any $A, B \in \mathscr{B}$, $A \subseteq B$, $B \in \mathscr{U}$, we have $A \in \mathscr{U}$,

(ii) $\mathscr{U}$ is *linearly ω-complete*, i.e. for any countable chain $a \subseteq \mathscr{U}$ we have $\bigcup a \in \mathscr{U}$.

A quasi-ideal $\mathscr{U}$ is called *proper*, if

(iii) $\emptyset \neq \mathscr{U} \neq \mathscr{B}$, i.e. $\emptyset \in \mathscr{U}$, $X \notin \mathscr{U}$.

A system $\boldsymbol{U}$ of ω-complete quasi-ideals in $\mathscr{B}$ is called *ω-quasi-maximal*, if

(iv) for any quasi-ideal $\mathscr{U} \in \boldsymbol{U}$ there is a countable subset $\boldsymbol{a} \subseteq \boldsymbol{U}$ such that for any $A \in \mathscr{B}$, $A \notin \mathscr{U} \equiv -A \in \bigcup \boldsymbol{a}$ holds true.

The system $\boldsymbol{U}$ is called *closed*, if

(v) for any finite subset $\boldsymbol{a} \subseteq \boldsymbol{U}$, we have $\bigcup \boldsymbol{a} \in \boldsymbol{U}$, $\bigcap \boldsymbol{a} \in \boldsymbol{U}$.

It is easy to verify that $\mathscr{L} = \{\boldsymbol{L}^r;\ r \in Q,\ 0 \leqslant r < 1\}$ is a closed ω-quasi-maximal system of proper ω-complete quasi-ideals in $\mathscr{B}(\boldsymbol{I})$. The system $\mathscr{K}$

$=\{K^J;\ J\in \boldsymbol{J}\}$ is also ω-quasi-maximal and its elements are proper ω-complete ideals. $\mathscr{K}$ itself is not closed but $\mathscr{K}$ can be extended to a closed system $\bar{\mathscr{K}}$, according to the following theorem.

THEOREM 1.1. *If* $\boldsymbol{U}$ *is a* ω*-quasi-maximal system of proper* ω*-complete quasi-ideals in* $\mathscr{B}$, *then*

$$\bar{\boldsymbol{U}}=\{\bigcup\{\bigcap d;\ d\in\mathscr{D}\};\ \mathscr{D}\in P^{<\omega}(P^{<\omega}(\boldsymbol{U}))\}$$

is the least closed ω*-quasi-maximal system of proper* ω*-complete quasi-ideals in* $\mathscr{B}$ *extending* $\boldsymbol{U}$.

Proof. Straightforward. □

2. Iterated products. To provide the condition (**) mentioned above, which is necessary for the definition of the product of ideals, we shall use a wider notion, that of quasi-ideal. The product of quasi-ideals will be defined in the same way as for ideals.

Let X, X_0, X_1, $\mathscr{B}$, $\mathscr{B}_0$, $\mathscr{B}_1$ have the same meaning as in the previous section and let $\mathscr{U}_0$, $\mathscr{U}_1$ be quasi-ideals in $\mathscr{B}_0$, $\mathscr{B}_1$, respectively. Assuming the conditions (*) and (**) (the later with $\mathscr{U}_1$ instead of I_1) we define the product $\mathscr{U}_0\times\mathscr{U}_1$ as follows:

$$A\notin\mathscr{U}_0\times\mathscr{U}_1\equiv\{x_0\in X_0;\ x_0A\notin\mathscr{U}_1\}\notin\mathscr{U}_0,\quad \text{for}\quad A\in\mathscr{B}.$$

Our first task will be to verify that the product of quasi-ideals preserves ω-completeness and ω-quasi-maximal systems.

THEOREM 2.1. *If* $\boldsymbol{U}_0$, $\boldsymbol{U}_1$ *are* ω*-quasi-maximal systems of proper* ω*-complete quasi-ideals in* $\mathscr{B}_0$, $\mathscr{B}_1$, *such that the conditions* (*) *and* (**) *for any* $\mathscr{U}_1\in\boldsymbol{U}_1$ *are satisfied, and if, moreover,* $\boldsymbol{U}_1$ *is closed, then* $\boldsymbol{U}=\{\mathscr{U}_0\times\times\mathscr{U}_1;\ \mathscr{U}_0\in\boldsymbol{U}_0,\ \mathscr{U}_1\in\boldsymbol{U}_1\}$ *is a* ω*-quasi-maximal system of proper* ω*-complete quasi-ideals in* $\mathscr{B}$.

Proof. It is easy to check the following monotony rules for A, $B\in\mathscr{B}$, $a\subseteq\mathscr{B}$, $x_0\in X_0$, $\mathscr{U}_1\in\boldsymbol{U}_1$: if $A\subseteq B$, then $x_0A\subseteq x_0B$, $A\mathscr{U}_1\subseteq B\mathscr{U}_1$; if a is a chain in $\mathscr{B}$, then $\{x_0A;\ A\in a\}$, $\{A\mathscr{U}_1;\ A\in a\}$ are chains in $\mathscr{B}_1$, $\mathscr{B}_0$, respectively.

Using this monotony, we can show that any $\mathscr{U}\in\boldsymbol{U}$ has the properties (i), (ii), (iii). To verify (iv), we bring a lemma.

LEMMA. *If* $\mathscr{U}$ *is a* ω*-complete quasi-ideal in a field* $\mathscr{B}$ *and if* $a\subseteq\mathscr{B}$ *is countable, then*

$$\bigcup a\notin\mathscr{U}\equiv(\exists a'\in P^{<\omega}(a))\bigcup a'\notin\mathscr{U}.$$

Proof. Use the ω-completeness of $\mathscr{U}$ and the fact that

$$\bigcup a=\bigcup\{A_i;\ i\in\omega\}=\bigcup\{A_0\cup\ldots\cup A_i;\ i\in\omega\}.\ \square$$

We are prepared to prove the ω-maximality (iv) of $\boldsymbol{U}$. As $\boldsymbol{U}_0$, $\boldsymbol{U}_1$ are ω-

quasi-maximal, for any given $\mathscr{U}_0 \in \boldsymbol{U}_0$, $\mathscr{U}_1 \in \boldsymbol{U}_1$ there are countable sets $\boldsymbol{a}_0 \subseteq \boldsymbol{U}_0$, $\boldsymbol{a}_1 \subseteq \boldsymbol{U}_1$ such that for any $A_0 \in \mathscr{B}_0$, $A_1 \in \mathscr{B}_1$ we have

$$A_0 \notin \mathscr{U}_0 \equiv -A_0 \in \bigcup \boldsymbol{a}_0, \quad A_1 \notin \mathscr{U}_1 \equiv -A_1 \in \bigcup \boldsymbol{a}_1.$$

Now, for $A \in \mathscr{B}$, we have the following equivalent statements:

$$\begin{aligned}
&A \notin \mathscr{U}_0 \times \mathscr{U}_1,\\
&\{x_0 \in X_0;\ x_0 A \notin \mathscr{U}_1\} \notin \mathscr{U}_0,\\
&\{x_0 \in X_0;\ x_0(-A) \in \bigcup \boldsymbol{a}_1\} \notin \mathscr{U}_0,\\
&\bigcup \{-(-A)\mathscr{V}_1;\ \mathscr{V}_1 \in \boldsymbol{a}_1\} \notin \mathscr{U}_0,\\
&(\exists \boldsymbol{a}' \in P^{<\omega}(\boldsymbol{a}_1)) \bigcup \{-(-A)\mathscr{V}_1;\ \mathscr{V}_1 \in \boldsymbol{a}'\} \notin \mathscr{U}_0,\\
&(\exists \boldsymbol{a}' \in P^{<\omega}(\boldsymbol{a}_1)) \{x_0 \in X_0;\ x_0(-A) \in \bigcup \boldsymbol{a}'\} \notin \mathscr{U}_0,\\
&(\exists \boldsymbol{a}' \in P^{<\omega}(\boldsymbol{a}_1))(\exists \mathscr{V}_0 \in \boldsymbol{a}_0)((-A)(\bigcup \boldsymbol{a}')) \in \mathscr{V}_0,\\
&(\exists \boldsymbol{a}' \in P^{<\omega}(\boldsymbol{a}_1))(\exists \mathscr{V}_0 \in \boldsymbol{a}_0) -A \in \mathscr{V}_0 \times (\bigcup \boldsymbol{a}').
\end{aligned}$$

We have used lemma to replace the infinite countable set $\boldsymbol{a}_1$ by its finite subsets. For $\boldsymbol{a}' \in P^{<\omega}(\boldsymbol{a}_1)$ we have, as $\boldsymbol{U}_1$ is closed, $\bigcup \boldsymbol{a}' \in \boldsymbol{U}_1$. Therefore

$$\boldsymbol{a} = \{\mathscr{V}_0 \times (\bigcup \boldsymbol{a}');\ \mathscr{V}_0 \in \boldsymbol{a}_0,\ \boldsymbol{a}' \in P^{<\omega}(\boldsymbol{a}_1)\}$$

is a countable subset of $\boldsymbol{U}$ such that for any $A \in \mathscr{B}$, $A \notin \mathscr{U}_0 \times \mathscr{U}_1 \equiv -A \in \bigcup \boldsymbol{a}$ holds true. □

Remark. It is easy to verify that if $\mathscr{U}_0 \in \boldsymbol{U}_0$, $\mathscr{U}_1 \in \boldsymbol{U}_1$ are, moreover, ideals, then $\mathscr{U}_0 \times \mathscr{U}_1$ is a (ω-complete) ideal.

The next theorem guarantees the conditions (*) and (**).

THEOREM 2.2. *If X_0, X_1, $X = X_0 \times X_1$ are topological spaces (X with the product topology), $\mathscr{B}_0$, $\mathscr{B}_1$, $\mathscr{B}$ are their fields of all Borel subsets and if $\boldsymbol{U}_1$ is a ω-quasi-maximal system of proper ω-complete quasi-ideals in $\mathscr{B}_1$, then the conditions (*) and (**) for any $\mathscr{U}_1 \in \boldsymbol{U}_1$ are fulfilled.*

Proof. We shall follow the idea from the previous section. Let us take a countable linearly ordered subset $a \subseteq \mathscr{B}$ such that for any $A \in a$ or $-A \in a$ the condition (**) is satisfied for any $\mathscr{U}_1 \in \boldsymbol{U}_1$. Then, the ω-completeness of $\mathscr{U}_1 \in \boldsymbol{U}_1$ gives $(\bigcup a) = \bigcup \{A\mathscr{U}_1;\ A \in a\} \in \mathscr{B}_0$. To prove $(\bigcap a)\mathscr{U}_1 \in \mathscr{B}_0$ we shall use the ω-quasi-maximality of $\boldsymbol{U}_1$: $\mathscr{U}_1$ being fixed, we take a countable subset $\boldsymbol{a} \subseteq \boldsymbol{U}_1$ such that for any $A_1 \in \mathscr{B}_1$, $A_1 \notin \mathscr{U}_1 \equiv -A_1 \in \bigcup \boldsymbol{a}$ holds true. We have

$$\begin{aligned}
(\bigcap a)\mathscr{U}_1 &= \{x_0 \in X_0;\ x_0(\bigcap a) \notin \mathscr{U}_1\}\\
&= \{x_0 \in X_0;\ -x_0(\bigcap a) \in \bigcup \boldsymbol{a}\}\\
&= \bigcup \{\{x_0 \in X_0;\ -x_0(\bigcap a) \in \mathscr{V}\};\ \mathscr{V} \in \boldsymbol{a}\}\\
&= \bigcup \{\{x_0 \in X_0;\ \bigcup \{x_0(-A);\ A \in a\} \in \mathscr{V}\};\ \mathscr{V} \in \boldsymbol{a}\}\\
&= \bigcup \{-\bigcup \{(-A)\mathscr{V};\ A \in a\};\ \mathscr{V} \in \boldsymbol{a}\} \in \mathscr{B}_0. \ \square
\end{aligned}$$

The Theorems 2.1 and 2.2 make it possible to construct the product of any finite number of quasi-ideals (or, which is of main importance, of ideals), provided that the factors are members of closed ω-quasi-maximal systems. It is easy to verify the associativity of product, therefore the product of n factors can be denoted simply as $\mathcal{U}_0 \times \mathcal{U}_1 \times \ldots \times \mathcal{U}_{n-1}$, without brackets. It will be useful to get some information on the saturatedness of the product. First we define some related notions.

A quasi-ideal $\mathcal{U}$ in a field $\mathcal{B}$ is called *(i, j)-quasi-saturated* if any subset $a \subseteq \mathcal{B}$ of cardinality $\operatorname{card} a \geqslant j$ with elements $A \in a$ fulfilling $-A \in \mathcal{U}$, contains a subset $a' \subseteq a$, $\operatorname{card} a' \geqslant i$ such that $\bigcap a'$ is non-empty.

$\mathcal{U}$ is called *ω-quasi-saturated*, if for any $i \in \omega$ there is $j \in \omega$ such that $\mathcal{U}$ is (i, j)-quasi-saturated.

THEOREM 2.3. *If $U_0, U_1, \ldots, U_{n-1}$ are closed ω-quasi-maximal systems of proper ω-complete ω-quasi-saturated quasi-ideals, then for any $\mathcal{U}_0 \in U_0$, $\mathcal{U}_1 \in U_1, \ldots, \mathcal{U}_{n-1} \in U_{n-1}$ the product $\mathcal{U}_0 \times \mathcal{U}_1 \times \ldots \times \mathcal{U}_{n-1}$ is ω-quasi-saturated. If, moreower, $\mathcal{U}_0 \times \mathcal{U}_1 \times \ldots \times \mathcal{U}_{n-1}$ is an ideal, then it is ω-saturated.*

Proof. For the first part of the theorem it is sufficient to consider two factors, say $\mathcal{U}_0 \in U_0$, $\mathcal{U}_1 \in U_1$. Clearly, if $\mathcal{U}_0$ is (i, j)-quasi-saturated and $\mathcal{U}_1$ is (h, i)-quasi-saturated, then $\mathcal{U}_0 \times \mathcal{U}_1$ is (h, j)-quasi-saturated.

Therefore, if $\mathcal{U}_0$, $\mathcal{U}_1$ are ω-quasi-saturated, so is $\mathcal{U}_0 \times \mathcal{U}_1$.

To prove the second part, we show that if U is a ω-quasi-maximal system of ω-complete ω-quasi-saturated quasi-ideal in a field $\mathcal{B}$, then any ideal $\mathcal{U} \in U$ is ω-saturated.

Let us take an ideal $\mathcal{U} \in U$ and an uncountable set $a \subseteq \mathcal{B}$ of elements not belonging to $\mathcal{U}$, such that $A \cap B$ belongs to $\mathcal{U}$ for any $A, B \in a$, $A \neq B$. By the ω-completeness of $\mathcal{U}$, we may assume that elements of a are pairwise disjoint. The ω-quasi-maximality of U implies the existence of a countable $\boldsymbol{a} \subseteq U$ such that for any $A \in \mathcal{B}$, $A \notin \mathcal{U} \equiv -A \in \bigcup \boldsymbol{a}$ holds. Thus, there is $\mathcal{V} \in \boldsymbol{a}$, such that $-A \in \mathcal{V}$ is true for infinitely many $A \in \boldsymbol{a}$. It is in contradiction with the fact that the quasi-ideal $\mathcal{V}$ is ω-quasi-saturated (and, therefore, $(2, j)$-quasi-saturated for some $j \in \omega$).

THEOREM 2.4. *Elements of systems $\mathcal{L}$, $\bar{\mathcal{K}}$ are ω-quasi-saturated.*

Proof. Elements of $\mathcal{L}$ are of the form L^r, for r rational, $0 \leqslant r < 1$. For any such r, there is natural $h > 0$ such that $r < 1 - 1/h$. Then, L^r is (i, ih)-quasi-saturated for any natural $i > 0$.

Elements of $\mathcal{K}$, and their finite intersections as well, are proper ideals and are, therefore, (i, i)-quasi-saturated for any natural $i > 0$. It is easy to verify then, that any element of $\bar{\mathcal{K}}$, of the form $\bigcup \{\bigcap d;\ d \in \mathcal{D}\}$ for $\mathcal{D} \in P^{<\omega}(P^{<\omega}(\mathcal{K}))$, is, in denotation, $\operatorname{card} \mathcal{D} = h$, (i, ih)-quasi-saturated for any natural $i > 0$. $\square$

As a consequence of our previous considerations we have the following theorem which involves the description of iterated products of ideals L, K.

THEOREM 2.5. *If every of quasi-ideals $\mathscr{U}_0, \mathscr{U}_1, \ldots, \mathscr{U}_{n-1}$ belongs to $\mathscr{L}$ or to $\mathscr{K}$, then the product $\mathscr{U}_0 \times \mathscr{U}_1 \times \ldots \times \mathscr{U}_{n-1}$ is a ω-complete ω-quasi-saturated quasi-ideal. If, moreover, $\mathscr{U}_0, \mathscr{U}_1, \ldots, \mathscr{U}_{n-1}$ are ideals, then $\mathscr{U}_0 \times \mathscr{U}_1 \times \ldots \times \mathscr{U}_{n-1}$ is a ω-complete ω-saturated ideal.*

3. Complete Boolean products of $\mathscr{C}$, $\mathscr{R}$. Products of ideals L, K defined in the first two sections enable us to define interesting examples of complete Boolean product of algebras $\mathscr{C}$, $\mathscr{R}$.

By a *complete Boolean product* of Boolean algebras $\mathscr{A}_0$, $\mathscr{A}_1$ we understand a triple $(i_0, i_1, \mathscr{B})$, where

(a) $\mathscr{B}$ is a complete Boolean algebra,

(b) $i_0\colon \mathscr{A}_0 \to \mathscr{B}$, $i_1\colon \mathscr{A}_1 \to \mathscr{B}$ are complete injections,

(c) $i_0(\mathscr{A}_0) \cup i_1(\mathscr{A}_1)$ completely generates $\mathscr{B}$.

If, moreover,

(d) $i_0(\mathscr{A}_0)$, $i_1(\mathscr{A}_1)$ are independent subalgebras in $\mathscr{B}$, then the product $(i_0, i_1, \mathscr{B})$ is said to be *independent*.

Sometimes, properties of products other than independence are investigated. For Boolean-valued models of the set theory ZFC, the local independence and the local disjointness are important; see [1], [2].

The product $(i_0, i_1, \mathscr{B})$ is called *locally independent*, or *locally disjoint*, if the set $\{\mathscr{V} \in \mathscr{B};\ i_0(\mathscr{A}_0)|\mathscr{V},\ i_1(\mathscr{A}_1)|\mathscr{V}$ are independent in $\mathscr{B}|\mathscr{V}\}$, or the set $\{\mathscr{V} \in \mathscr{B};\ i_0(\mathscr{A}_0)|\mathscr{V} \cap i_1(\mathscr{A}_1)|\mathscr{V} = \{0, \mathscr{V}\}\}$, respectively, is dense in $\mathscr{B}$.

It is proved in [2] that for fixed $\mathscr{A}_0$, $\mathscr{A}_1$ there is exactly one (up to isomorphism) such complete independent Boolean product of $\mathscr{A}_0$, $\mathscr{A}_1$ that is locally independent. This product is minimal in the ordering of products defined as follows: $(i'_0, i'_1, \mathscr{B}') \leqslant (i_0, i_1, \mathscr{B}) \equiv$ there exists complete homomorphism $h\colon \mathscr{B} \to \mathscr{B}'$ such that $i'_0 = hi_0$, $i'_1 = hi_1$. The minimal product is characterized by the property that the set $\{i_0(A_0) \wedge i_1(A_1);\ A_0 \in \mathscr{A}_0,\ A_1 \in \mathscr{A}_1\}$ is dense in $\mathscr{B}$.

In the contrary to the uniqueness of the locally independent products, we show in this section that it is possible to construct infinitely many locally disjoint complete independent Boolean products of algebras $\mathscr{C}$, $\mathscr{R}$. The construction, in a more general form, is described in Theorems 3.1 and 3.2.

THEOREM 3.1. *Let C_0, C_1 be ω-complete fields of subsets of spaces Y_0, Y_1, let J_0, J_1 be proper ω-complete ω-saturated ideals in C_0, C_1, let C_0, C_1 ω-generate the field C in the Cartesian product $Y = Y_0 \times Y_1$. If J is a proper ω-complete ω-saturated ideal in C, such that*

(***) *for any $A_0 \in C_0$, $A_1 \in C_1$, $A_0 \times A_1 \notin J \equiv A_0 \notin J_0$, $A_1 \notin J_1$ holds true,*

then, denoting $i_0^J(A/J_0) = (A \times Y_1)/J$ for any $A \in C_0$, $i_1^J(A/J_1) = (Y_0 \times A)/J$ for any $A \in C_1$, we have

(a) $(i_0^J, i_1^J, C/J)$ *is a complete independent Boolean product of algebras* C_0/J_0, C_1/J_1,

(b) *for any proper* ω*-complete* ω*-saturated ideal* I *in* C *fulfilling* (***), $(i_0^I, i_1^I, C/I) \leqslant (i_0^J, i_1^J, C/J)$ *is equivalent to* $I \supseteq J$.

Proof. (a) Using (***), it is easy to verify that $i_0^J\colon C_0/J_0 \to C/J$, $i_1^J\colon C_1/J_1 \to C/J$ are uniquely defined ω-complete injections and that $i_0^J(C_0/J_0)$, $i_1^J(C_1/J_1)$ are independent and ω-generating in C/J. By ω-saturatedness of J_0, J_1, J, algebras C_0/J_0, C_1/J_1, C/J and injections i_0^J, i_1^J are complete.

(b) If $h\colon C/J \to C/I$ is a ω-complete homomorphism such that $i_0^I = hi_0^J$, $i_1^I = hi_1^J$ holds, then, for $A_0 \in C_0$, $A_1 \in C_1$, we have $h((A_0 \times A_1)/J) = (A_0 \times A_1)/I$. As C_0, C_1 ω-generate C, it must be $h(A/J) = A/I$ for any $A \in C$. That gives $J \subseteq I$. The converse implication in (b) is clear. □

In the following, we need the notion of (α, β)-tree: if α, β are ordinal numbers and C is a system of sets, then a function A is called an *(α, β)-tree* in C if $D(A) \subseteq {}^{<\alpha}\beta$, $W(A) \subseteq C$ and if for any φ, $\psi \in {}^{<\alpha}\beta$ the implications $\varphi \subseteq \psi$, $\psi \in D(A) \to \varphi \in D(A)$, $A(\varphi) \supseteq A(\psi)$, $\varphi \nsubseteq \psi$, $\psi \nsubseteq \varphi$, $\varphi, \psi \in D(A) \to A(\varphi) \cap A(\psi) = \emptyset$ hold true. The well-known Suslin's hypothesis says, that if in a tree A any chain and any antichain is countable, then A itself is countable. Suslin's hypothesis is independent on the axioms of ZFC (see [4], [8], [7]).

THEOREM 3.2. *Let ideals* J_0, J_1, J *in fields* C_0, C_1, C *fulfill the conditions of Theorem* 3.1 *and the condition*

(****) *for any* $(\omega+1, \omega)$*-trees* A_0 *in* C_0, A_1 *in* C_1 *the implication*

$$(\forall \psi \in {}^{\omega}\omega)\, A_0(\psi) \times A_1(\psi) \in J \;\to\; \bigcup \{A_0(\psi) \times \dot{A}_1(\psi);\ \psi \in {}^{\omega}\omega\} \in J$$

holds true.

Then, assuming the Suslin's hypothesis, the Boolean product $(i_0^J, i_1^J, C/J)$ *is locally disjoint.*

Proof. We proceed by contradiction and assume that:

(A) There is a non-zero element $\bar{\mathscr{U}} \in C/J$ such that for any non-zero element $\bar{\mathscr{V}} \in C/J$, $\bar{\mathscr{V}} \leqslant \bar{\mathscr{U}}$ there exist elements $\bar{A}_0 \in C_0/J_0$, $\bar{A}_1 \in C_1/J_1$ such that $0 < i_0^J(\bar{A}_0) \wedge \bar{\mathscr{V}} = i_1^J(\bar{A}_1) \wedge \bar{\mathscr{V}} < \bar{\mathscr{V}}$.

We can take an element $\mathscr{U} \in C$ such that $[\mathscr{U}]_J = \bar{\mathscr{U}}$ fulfills (A) and define two $(\omega_1, 2)$-trees, A_0 in C_0, A_1 in C_1 with the same domain $D = D(A_0) = D(A_1) \subseteq {}^{<\omega_1}2$, in such a way that the $(\omega_1, 2)$-tree A in C, defined by $A(\varphi) = A_0(\varphi) \times A_1(\varphi)$ for any $\varphi \in D$, has the following properties (for $\alpha \in \omega_1$ we write $D(\alpha)$ instead of $D \cap {}^{\alpha}2$):

(i) $(\forall \varphi \in D)\, \mathscr{U} \cap A(\varphi) \notin J$;

(ii) $(\forall \alpha \in \omega_1)\, \mathscr{U} - \bigcup \{A(\varphi);\ \varphi \in D(\alpha)\} \in J$;

(iii) $(\forall \varphi \in D)\, \varphi\hat{0},\, \varphi\hat{1} \in D$;

(iv) $(\forall \alpha \in \omega_1)\, 0 < \operatorname{card} D(\alpha) \leqslant \omega$.

The trees A_0, A_1 are defined on $D(\alpha)$ by induction with respect to $\alpha \in \omega_1$. On non-limit steps we use the assumption (A), on limit steps we take the intersections of the branches such that (i) is fulfilled. The crucial point of the proof is to verify (ii) at limit steps; the conditions (iii), (iv) are clear.

Let $\lambda \in \omega_1$ be limit ordinal. We fix an increasing sequence $(\lambda_n;\ n \in \omega)$ converging to λ. We denote for $k \in 2$, $n \in \omega$

$$\bar{A}_k = \bigcup \{A_k(\varphi);\ \varphi \in D(\lambda)\}, \quad \bar{A} = \bigcup \{A(\varphi);\ \varphi \in D(\lambda)\},$$
$$A^{(n)} = \bigcup \{A(\varphi);\ \varphi \in D(\lambda_n)\}, \quad A^{(\omega)} = \bigcap \{A^{(n)};\ n \in \omega\}.$$

For $n < m < \omega$ we have $A^{(n)} \supseteq A^{(m)} \supseteq A^{(\omega)} \supseteq \bar{A}$ and, therefore

$$\begin{aligned}\mathscr{U} - \bar{A} &= (\mathscr{U} - A^{(\omega)}) \cup (\mathscr{U} \cap A^{(\omega)} - \bar{A}) \\ &= (\mathscr{U} - A^{(\omega)}) \cup \big(\mathscr{U} \cap A^{(\omega)} \cap -\bar{A} \cap (\bar{A}_0 \times \bar{A}_1)\big) \cup \\ &\qquad \cup \big(\mathscr{U} \cap A^{(\omega)} \cap -\bar{A} \cap -(\bar{A}_0 \times \bar{A}_1)\big).\end{aligned}$$

It is easy to show that the first two summands of the union belong to J. For the third summand we have:

$$\begin{aligned}&\mathscr{U} \cap A^{(\omega)} \cap -\bar{A} \cap -(\bar{A}_0 \times \bar{A}_1) \\ &\quad \subseteq \mathscr{U} \cap A^{(\omega)} \cap -(\bar{A}_0 \times \bar{A}_1) \\ &\quad = \bigcap \{\mathscr{U} \cap A^{(n)} \cap -(\bar{A}_0 \times \bar{A}_1);\ n \in \omega\} \\ &\quad = \bigcap \{\bigcup \{\mathscr{U} \cap A(\varphi) \cap -(\bar{A}_0 \times \bar{A}_1);\ \varphi \in D(\lambda_n)\};\ n \in \omega\} \\ &\quad = \bigcup \{\bigcap \{\mathscr{U} \cap A(\varepsilon_n) \cap -(\bar{A}_0 \times \bar{A}_1);\ n \in \omega\};\ \varepsilon \in \prod \{D(\lambda_n);\ n \in \omega\}\} \\ &\quad \subseteq \bigcup \{\mathscr{U} \cap A(\varphi) \cap -(\bar{A}_0 \times \bar{A}_1);\ \varphi \in {}^{\lambda}2\}.\end{aligned}$$

For $\varphi \notin D(\lambda)$, $\mathscr{U} \cap A(\varphi) \in J$ and for $\varphi \in D(\lambda)$,

$$A(\varphi) \cap -(\bar{A}_0 \times \bar{A}_1) = \big(A_0(\varphi) \times A_1(\varphi)\big) \cap -(\bar{A}_0 \times \bar{A}_1) = \emptyset \in J$$

hold. By the saturatedness of J and by (****) we get

$$\bigcup \{\mathscr{U} \cap \big(A_0(\varphi) \times A_1(\varphi)\big) \cap -(\bar{A}_0 \times \bar{A}_1);\ \varphi \in {}^{\lambda}2\} \in J.$$

By (ii), the tree A has non-zero elements at any level $\alpha \in \omega_1$, by ω-saturation of J, any antichain in A is countable. Therefore, Suslin's hypothesis implies that there is $\varphi \in {}^{\omega_1}2$ such that for any $\alpha \in \omega_1$, $\varphi|\alpha \in D$ holds. By (i), (iii), $\big(A(\varphi|\alpha)/J;\ \alpha \in \omega_1\big)$ is a strictly decreasing sequence in C/J, in contradiction to the ω-saturation of J. □

The general construction described above will be specified by using products of ideals in various order.

We shall assume that $U_0, U_1, \ldots, U_{n-1}$ are ω-quasi-maximal systems of proper ω-complete ω-quasi-saturated quasi-ideals in the fields $\mathscr{B}_0$,

$\mathcal{B}_1, \ldots, \mathcal{B}_{n-1}$ of all Borel subsets of topological spaces $X_0, X_1, \ldots, X_{n-1}$. For $0 \leqslant k < m \leqslant n$ we denote

$$X_{km} = X_k \times \ldots \times X_{m-1}, \qquad \mathcal{B}_{km} = \mathcal{B}(X_{km}),$$

$$U_{km} = \{\mathcal{U}_k \times \ldots \times \mathcal{U}_{m-1};\ \mathcal{U}_k \in U_k, \ldots, \mathcal{U}_{m-1} \in U_{m-1}\}.$$

Let $\alpha = (\alpha(0), \ldots, \alpha(n-1))$ be a permutation of the set $n = \{0, 1, \ldots, n-1\}$. For $x = (x_0, x_1, \ldots, x_{n-1}) \in X_0 \times \ldots \times X_{n-1} = X$, $A \subseteq X$, $\mathcal{U}_0 \in U_0, \ldots, \mathcal{U}_{n-1} \in U_{n-1}$ we denote

$$\alpha x = (x_{\alpha(0)}, \ldots, x_{\alpha(n-1)}) \in X_{\alpha(0)} \times \ldots \times X_{\alpha(n-1)} = \alpha X;$$

$$\alpha A = \{\alpha x;\ x \in A\}, \qquad \alpha \mathcal{U}_{0n} = \mathcal{U}_{\alpha(0)} \times \ldots \times \mathcal{U}_{\alpha(n-1)}.$$

The permutated product $\mathcal{U}^{\alpha}_{0n}$ of ideals $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_{n-1}$ in order α is defined as follows: for any $A \in \mathcal{B}_{0n}$,

$$A \in \mathcal{U}^{\alpha}_{0n} \equiv \alpha A \in \alpha \mathcal{U}_{0n}.$$

For $0 < l < n$ we denote by $\mathcal{U}^{\alpha}_{0l}$, $\mathcal{U}^{\alpha}_{ln}$ the projections of the ideal $\mathcal{U}^{\alpha}_{0n}$ into $\mathcal{B}_{0l}$, $\mathcal{B}_{ln}$, respectively, and by i^{α}_{0l}, i^{α}_{ln} the natural injections of $\mathcal{B}_{0l}/\mathcal{U}^{\alpha}_{0l}$, $\mathcal{B}_{ln}/\mathcal{U}^{\alpha}_{ln}$, respectively, into $\mathcal{B}_{0n}/\mathcal{U}^{\alpha}_{0n}$. In this denotation we have

THEOREM 3.3. *For $0 < l < n$, for any ideals $\mathcal{U}_0 \in U_0$, $\mathcal{U}_1 \in U_1, \ldots, \mathcal{U}_{n-1} \in U_{n-1}$ and for any permutation α, under the assumption of Suslin's hypothesis, the triple $(i^{\alpha}_{0l}, i^{\alpha}_{ln}, \mathcal{B}_{0n}/\mathcal{U}^{\alpha}_{0n})$ is a complete independent locally disjoint Boolean product of algebras $\mathcal{B}_{0l}/\mathcal{U}^{\alpha}_{0l}$, $\mathcal{B}_{ln}/\mathcal{U}^{\alpha}_{ln}$.*

Proof. By Theorems 3.1, 3.2, it is sufficient to verify conditions (***), (****). An intermediate verification by induction gives, for any $A_{0l} \in \mathcal{B}_{0l}$, $A_{ln} \in \mathcal{B}_{ln}$,

$$A_{0l} \times A_{ln} \notin \mathcal{U}^{\alpha}_{0n} \equiv A_{0l} \notin \mathcal{U}^{\alpha}_{0l}, \quad A_{ln} \notin \mathcal{U}^{\alpha}_{ln}.$$

The condition (****) will be also verified by induction. Let us assume that (****) is fulfilled in the case of less than n factors. Let T_0, T_1 be $(\omega+1, \omega)$-trees in $\mathcal{B}_{0l}$, $\mathcal{B}_{ln}$, respectively and let

$$\bigcup \{T_0(\psi) \times T_1(\psi);\ \psi \in {}^{\omega}\omega\} \notin \mathcal{U}^{\alpha}_{0n}$$

hold. By definition, we have

$$S = \{x_{\alpha(0)} \in X_{\alpha(0)};\ \bigcup \{x_{\alpha(0)}(T_0(\psi) \times T_1(\psi));\ \psi \in {}^{\omega}\omega\} \notin \mathcal{U}^{\alpha}_{1n}\} \notin \mathcal{U}_{\alpha(0)}.$$

We may assume $\alpha(0) < l$. (The case $l \leqslant \alpha(0)$ differs only by denotation.) For $l = 1$ (then $\alpha(0) = 0$ holds) and for $x_{\alpha(0)} \in T_0(\bar{\psi})$, $\bar{\psi} \in {}^{\omega}\omega$ we have

$$\bigcup \{x_{\alpha(0)}(T_0(\psi) \times T_1(\psi));\ \psi \in {}^{\omega}\omega\} = T_1(\bar{\psi}),$$

therefore

$$S = \{x_{\alpha(0)} \in X_{\alpha(0)};\ (\exists\psi \in {}^{\omega}\omega)\, x_{\alpha(0)} \in T_0(\psi),\ T_1(\psi) \notin \mathcal{U}^{\alpha}_{1n}\}$$
$$= \{x_{\alpha(0)} \in X_{\alpha(0)};\ (\exists\psi \in {}^{\omega}\omega)\big(x_{\alpha(0)}\ T_0(\psi)\big) \times T_1(\psi) \notin \mathcal{U}^{\alpha}_{1n}\} \notin \mathcal{U}_{\alpha(0)}.$$

The same result we get for $l > 1$, using the induction assumption. Further, by the ω-quasi-maximality of U^{α}_{1n} and the ω-completeness of $\mathcal{U}_{\alpha(0)}$, there exists $\mathcal{V} \in U^{\alpha}_{1n}$, such that

$$\{x_{\alpha(0)} \in X_{\alpha(0)};\ (\exists\psi \in {}^{\omega}\omega) - \big((x_{\alpha(0)}\ T_0(\psi)) \times T_1(\psi)\big) \in \mathcal{V}\} \notin \mathcal{U}_{\alpha(0)}$$

holds.

The quasi-ideal $\mathcal{V}$ is ω-quasi-saturated, so there exists $i \in \omega$, such that $\mathcal{V}$ is $(2, i)$-quasi-saturated. Then we cannot have more than i of functions $\psi \in {}^{\omega}\omega$, fulfilling $-\big((x_{\alpha(0)}\ T_0(\psi)) \times T_1(\psi)\big) \in \mathcal{V}$. Now, as $\mathcal{U}_{\alpha(0)}$ preserves the finite joins, there exists $\psi \in {}^{\omega}\omega$, such that

$$\{x_{\alpha(0)} \in X_{\alpha(0)};\ -\big((x_{\alpha(0)}\ T_0(\psi)) \times T_1(\psi)\big) \in \mathcal{V}\} \notin \mathcal{U}_{\alpha(0)},$$

i.e. $T_0(\psi) \times T_1(\psi) \notin \mathcal{U}^{\alpha}_{0n}$. The condition (****) is verified. □

In the final part of this section we apply the previous results to ideals $\boldsymbol{L}$, $\boldsymbol{K}$. We get

THEOREM 3.4. *For any $0 < l < n$, for the ideals $\mathcal{U}_0 = \cdots = \mathcal{U}_{l-1} = \boldsymbol{L}$, $\mathcal{U}_l = \cdots = \mathcal{U}_{n-1} = \boldsymbol{K}$ and for any permutation α, the triple $(i^{\alpha}_{0l}, i^{\alpha}_{ln}, \mathcal{B}(\boldsymbol{I}^n)/\mathcal{U}^{\alpha}_{0n})$ is a complete independent locally disjoint Boolean product of algebras $\mathcal{B}(\boldsymbol{I}^l)/\boldsymbol{L}^l$, $\mathcal{B}(\boldsymbol{I}^{n-l})/\boldsymbol{K}^{n-l}$.*

Proof. The assertion is a special case of Theorem 3.3 with Suslin's hypothesis left out. However, this point is clear, because the algebra $\mathcal{B}(\boldsymbol{I}^l)/\boldsymbol{L}^l$, being measurable, cannot contain any Suslin's tree. (Moreover, the algebra $\mathcal{B}(\boldsymbol{I}^{n-l})/\boldsymbol{K}^{n-l}$ having a countable base, cannot contain Suslin's tree, either). □

By the Fubini Theorem $\mathcal{B}(\boldsymbol{I}^l)/\boldsymbol{L}^l$ is isomorphic to $\mathcal{B}(\boldsymbol{I})/\boldsymbol{L} = \mathcal{R}$ and, analogously, $\mathcal{B}(\boldsymbol{I}^{n-l})/\boldsymbol{K}^{n-l}$ is isomorphic to $\mathcal{B}(\boldsymbol{I})/\boldsymbol{K} = \mathcal{C}$. Thus, Theorem 3.4 gives infinitely many locally disjoint products of $\mathcal{R}$, $\mathcal{C}$. Some of them are isomorphic, but we can find infinitely many non-isomorphic among them.

THEOREM 3.5. (a) *For any fixed natural n, the products $\boldsymbol{I}^{(k)}$ of ideals $\mathcal{U}_0 = \boldsymbol{L}$, $\mathcal{U}_1 = \ldots = \mathcal{U}_{n-1} = \boldsymbol{K}$ in order $\alpha_k = (1, 2, \ldots, k, 0, k+1, \ldots, n-1)$ for $k \in n$ are incomparable ideals in the field $\mathcal{B}(\boldsymbol{I}^n)$.*

(b) *For any natural n, there exist n pairwise incomparable complete independent locally disjoint Boolean products of algebras $\mathcal{R}$, $\mathcal{C}$.*

Proof. By the previous remark and by Theorems 3.1 and 3.2, it is sufficient to prove (a). We suppose $0 < k < m < n$ and we construct a set $A \in \mathcal{B}_{0n}$ such that $A \in \boldsymbol{I}^{(k)}$, $-A \in \boldsymbol{I}^{(m)}$ holds.

For any $j \in \omega$ we take a decomposition $r_j = \{N^{(j)}_i;\ i \in \omega\}$ of the set ω, such that for any $i \in \omega$, $\operatorname{card} N_i^{(j)} \geqslant i+j+2$ holds and for $j \leqslant l$ the decomposition r_j is a refinement of r_l. We set

$$A = \{x \in ({}^{\omega}2)^n;\ (\exists j \in \omega)(\forall i \in \omega)\, x_0 | N_i^{(j)} \neq x_{k+1} | N_i^{(j)}\}.$$

In the further we assume sequences from ${}^{\omega}2$ to be real numbers from I (in dyadic form).

We show first that $A \in I^{(k)}$ holds. It is easy to see that for any $j \in \omega$ the set

$$\{x_{k+1} \in {}^{\omega}2;\ (\forall i \in \omega)\, x_0 | N_i^{(j)} \neq x_{k+1} | N_i^{(j)}\}$$

is closed and nowhere dense in ${}^{\omega}2$. Therefore the set

$$\{x_{k+1} \in {}^{\omega}2;\ (\exists j \in \omega)(\forall i \in \omega)\, x_0 | N_i^{(j)} \neq x_{k+1} | N_i^{(j)}\}$$

belongs to $\boldsymbol{K}$. This implies that for any $(x_1, \ldots, x_k, x_0) \in ({}^{\omega}2)^{k+1}$ we have $(x_1, \ldots, x_k, x_0)\alpha_k A \in \boldsymbol{K}^{n-k-1}$ and

$$\alpha_k A \in \boldsymbol{K}^k \times \boldsymbol{L} \times \boldsymbol{K}^{n-k-1} = \alpha_k(\boldsymbol{L} \times \boldsymbol{K} \times \boldsymbol{K} \times \ldots \times \boldsymbol{K}),$$

i.e. $A \in \boldsymbol{I}^{(k)}$.

On the other hand, for any $i, j \in \omega$, $x_{k+1} \in {}^{\omega}2$ we have

$$\mu(\{x_0 \in {}^{\omega}2;\ x_0 | N_i^{(j)} = x_k | N_i^{(j)}\}) \leqslant 2^{-(i+j+2)},$$

which gives

$$\mu(\{x_0 \in {}^{\omega}2;\ (\forall j \in \omega)(\exists i \in \omega)\, x_0 | N_i^{(j)} = x_{k+1} | N_i^{(j)}\}) \leqslant \lim \{\textstyle\sum \{2^{-(i+j+2)};\ i \in \omega\};\ j \in \omega\} = 0.$$

This implies that for any $(x_1, \ldots, x_{k+1}, \ldots, x_m) \in ({}^{\omega}2)^m$ we have

$$(x_1, \ldots, x_{k+1}, \ldots, x_m)\alpha_m(-A) \in \boldsymbol{L} \times \boldsymbol{K}^{n-m-1}$$

and

$$\alpha_m(-A) \in \boldsymbol{K}^m \times \boldsymbol{L} \times \boldsymbol{K}^{n-m-1} = \alpha_m(\boldsymbol{L} \times \boldsymbol{K} \times \boldsymbol{K} \times \ldots \times \boldsymbol{K}),$$

i.e. $-A \in \boldsymbol{I}^{(m)}$. $\square$

THEOREM 3.6. (a) *The ideal* $I^{(0)}$ *is a maximal one in the set of all ideals* $\boldsymbol{I}$ *in the field* $\mathscr{B}_{0n}$, *fulfilling the condition: for any* $A_0 \in \mathscr{B}_0$, $A_{1n} \in \mathscr{B}_{1n}$,

$$A_0 \times A_{1n} \notin \boldsymbol{I} \equiv A_0 \notin \boldsymbol{L}, \quad A_{1n} \notin \boldsymbol{K}^{n-1}.$$

(b) *The minimal complete Boolean product of algebras* $\mathscr{R}$, $\mathscr{C}$ *is not the least product*, i.e. *there are complete independent Boolean products of* $\mathscr{R}$, $\mathscr{C}$, *incomparable with the minimal product.*

For any infinite cardinal $\mathfrak{m}$:

(c) *The minimal complete Boolean product of* $\mathscr{R}$, $\mathscr{C}$ *is their minimal* $\mathfrak{m}$-*product.*

(d) *The minimal* m-*product of* $\mathscr{R}$, $\mathscr{C}$ *is not their least* m-*product.*

Remark. The m-product of Boolean algebras, denoted as (m, 0)-product is described in [6]. There is a problem, whether the minimal (m, 0)-product is the least (m, 0)-product of given factors. The negative answer for factors $\mathscr{R}$, $\mathscr{R}$ is presented in [2].

Proof. The assertion (a) can be proved by induction in the form: for any $A \in \mathscr{B}_{0n}$, $A \notin I^{(0)}$ there are $A_0 \in \mathscr{B}_0$, $A_{1n} \in \mathscr{B}_{1n}$ such that $A_0 \notin L$, $A_{1n} \notin K^{n-1}$, $(A_0 \times A_{1n}) - A \in I^{(0)}$.

The main point of the induction is the following lemma.

LEMMA. *If* I *is* ω*-complete ideal in* $\mathscr{B}_0$, *then for any* $A \in \mathscr{B}_{01}$, $A \notin I \times K$ *there are* $A_0 \in \mathscr{B}_0$, $A_1 \in \mathscr{B}_1$ *such that* $A_0 \notin I$, $A_1 \in K$, $(A_0 \times A_1) - A \in I \times K$.

Proof of the lemma. By assumption we have $\{x_0 \in X_0;\ x_0 A \notin K\} \notin I$. From ω-quasi-maximality of $\mathscr{K}$ and from the ω-completeness of I, the existence of an interval J with rational endpoints follows, such that $A_0 = \{x_0 \in X_0;\ x_0(-A) \in K^J\} \notin I$. It suffices then to set $A_1 = J$. □

The assertions (b), (c), (d) are consequences of (a). □

Remark. The Boolean products involve the product algebra and the injections of the factors. In comparing the products, the injections play a substantial role. For example, the products $I^{(k)}$, $k = 0, 1, \ldots, n-1$ of ideals $\mathscr{U}_0 = L$, $\mathscr{U}_1 = \ldots = \mathscr{U}_{n-1} = K$, described in Theorem 3.5, induce n incomparable products of algebras $\mathscr{R}$, $\mathscr{C}$, which correspond to products of ideals $K^k \times L \times K^{n-k-1}$, $k = 0, 1, \ldots, n-1$. However, by the analogue of Fubini Theorem for category, among the above ideals we have not more than three substantially different ones: $L \times K$, $K \times L \times K$, $K \times L$. The question is, whether any two of these three ideals do not give isomorphic factor algebras.

Problem (**P 1285**). Are any two of Boolean algebras $\mathscr{B}(I^2)/(L \times K)$, $\mathscr{B}(I^3)/(K \times L \times K)$, $\mathscr{B}(I^2)/(K \times L)$ isomorphic? Analogously for algebras $\mathscr{B}(I^3)/(L \times K \times L)$, $\mathscr{B}(I^4)/(K \times L \times K \times L)$ and similar ones.

Remark. As I was informed by J. Cichoń, J. Truss recently proved that the algebras $\mathscr{B}(I^2)/(L \times K)$, $\mathscr{B}(I^2)/(K \times L)$ are not isomorphic.

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