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ON A REPRESENTATION OF GROUPOIDS AS SUMS OF DIRECTED SYSTEMS

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In [3] J. Płonka characterized the representation of an algebra as the sum of a directed system of algebras by the existence of a binary operation of this algebra, called *partition function*. Moreover, he found out that the sum of a directed system of algebras satisfies all regular equations which are true in the components and no other ones. He proved in [4] that under some additional assumptions one can represent the algebras in the equational class of all regular consequences of a given axiom system Σ as sums of directed systems of algebras of the equational class given by Σ.

In this paper we shall study some classes of idempotent groupoids which are defined by a finite number of regular equations and are representable as sums of directed systems of groupoids.

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First, we recall some notions of the above-mentioned papers which we need in the sequel. Let $\big(I, (\boldsymbol{A}_i)_{i\in I}, (\varphi_{ij})_{i,j\in I, i\leqslant j}\big)$ be a directed system of similar algebras, indexed by a join semilattice I and let A_i be the carrier set of the algebra $\boldsymbol{A}_i$ and $(f_t^i)_{t\in T}$ its algebraic structure. The φ_{ij} are homomorphisms from $\boldsymbol{A}_i$ to $\boldsymbol{A}_j$ for $i\leqslant j$, where, for $i=j$, $\varphi_{ij}=\mathrm{id}_{A_i}$ is the identity map of A_i, and, for $i\leqslant j\leqslant k$, $\varphi_{jk}\circ\varphi_{ij}=\varphi_{ik}$. Then, the sum $\boldsymbol{A}$ of this directed system is defined as an algebra of the same type, the carrier set A of $\boldsymbol{A}$ is the disjoint union of all carrier sets A_i, $i\in I$, and its algebraic structure is given by

$$f_t(a_1, \ldots, a_n) = f_t^{i_0}\big(\varphi_{i_1i_0}(a_1), \ldots, \varphi_{i_ni_0}(a_n)\big),$$

where $a_j\in A_{i_j}$ and i_0 is the least upper bound of all i_j, $j\in\{1, \ldots, n\}$. Here all algebras have finitary operations but no nullary ones.

An equation $f=g$ of an algebra is called *regular* iff all occurring variables of the terms f and g are exactly the same.

If $A = (A, (f_t)_{t\in T})$ is an arbitrary algebra of type $(n_t)_{t\in T}$, all n_t natural numbers, we call a binary operation $\circ$ of A a *partition function* or, shortly, *P-function* of A iff the following conditions are satisfied:

1.1. $(x\circ y)\circ z = x\circ(y\circ z)$,

1.2. $x\circ x = x$,

1.3. $x\circ(y\circ z) = x\circ(z\circ y)$,

1.4. $f_t(x_1, \ldots, x_{n_t})\circ y = f_t(x_1\circ y, \ldots, x_{n_t}\circ y)$ for all $t\in T$,

1.5. $y\circ f_t(x_1, \ldots, x_{n_t}) = y\circ f_t(y\circ x_1, \ldots, y\circ x_{n_t})$ for all $t\in T$,

1.6. $f_t(x_1, \ldots, x_{n_t})\circ x_k = f_t(x_1, \ldots, x_{n_t})$ for $t\in T$ and $1 \leqslant k \leqslant n_t$,

1.7. $y\circ f_t(y, \ldots, y) = y$ for all $t\in T$.

Let $\mathfrak{K}$ be an equational class defined by regular equations and let A be an algebra in $\mathfrak{K}$. Let $g(x, y)$ be a term of A and let $\mathfrak{K}^*$ be the equational class obtained from $\mathfrak{K}$ by adding the equation $g(x, y) = x$. Then, theorem III of [3] states

THEOREM 1. *The term $g(x, y)$ defines a P-function of A iff A is representable as the sum of a directed system of algebras of $\mathfrak{K}^*$.*

We shall apply this theorem to classes $\mathfrak{G}_{k,n}$ of groupoids $G = (G, \cdot)$, that means *algebras of type* 2, where k and n are natural numbers satisfying $2 \leqslant k$. For fixed k and n, $\mathfrak{G}_{k,n}$ is defined by the equations

2.1. $x_0\cdot x_1\cdot \ldots \cdot x_{k-1}\cdot x_k = x_k\cdot x_1\cdot \ldots \cdot x_{k-1}\cdot x_0$,

2.2. $(x\cdot y)\cdot \underbrace{z\cdot \ldots \cdot z}_{n\text{ times}} = x\cdot (y\cdot \underbrace{z\cdot \ldots \cdot z}_{n\text{ times}})$,

2.3. $x\cdot x = x$.

Here the terms without brackets stand for terms where all brackets are closed to the left. We shall use this notation throughout the whole paper. Furthermore, for abbreviation, we shall omit the dot and write xy instead of $x\cdot y$ and use n instead of n times and omit the bracket.

THEOREM 2. *Let G be a groupoid of $\mathfrak{G}_{k,n}$. Then the binary operation $\circ$ defined by*

$$x\circ y = xy\underset{n}{\ldots}y$$

is a P-function of G.

Before we prove the theorem, we give the following

LEMMA 3. *Let G be a groupoid of $\mathfrak{G}_{k,n}$. Then the following equations are satisfied:*

3.1. $(xy)(uv) = (xu)(yv)$,

3.2. $(xy)z = (xz)(yz)$,

3.3. $x(yz) = (xy)(xz)$,

3.4. $yx = xy\underset{k}{\ldots}y$,

3.5. $yx = x\underbrace{y \dots y}_{k+n}$,

3.6. $x\underbrace{y \dots y}_{n} = x(x\underbrace{y \dots y}_{n})$,

3.7. $x\underbrace{y \dots y}_{n} = x\underbrace{y \dots y}_{ln}$ for all natural numbers $l \geqslant 1$.

Proof.

3.1. $$\begin{aligned} (xy)(uv) &= (\underbrace{x \dots x}_{k-1}y)(uv) && \text{by } 2.3 \\ &= (uv\underbrace{x \dots x}_{k-2}y)x && \text{by } 2.1 \\ &= (yv\underbrace{x \dots x}_{k-2}u)x && \text{by } 2.1 \\ &= (\underbrace{x \dots x}_{k-1}u)(yv) && \text{by } 2.1 \\ &= (xu)(yv). \end{aligned}$$

3.2. $(xy)z = (xy)(zz) = (xz)(yz)$ by 2.3 and 3.1.

3.3. $x(yz) = (xx)(yz) = (xy)(xz)$ by 2.3 and 3.1.

3.4. $x\underbrace{y \dots y}_{k} = \underbrace{y \dots y}_{k}x = yx$ by 2.1 and 2.3.

3.5. $x\underbrace{y \dots y}_{n+k} = (x\underbrace{y \dots y}_{k-1})(\underbrace{y \dots y}_{n+1}) = x\underbrace{y \dots y}_{k} = yx$ by 2.2, 2.3 and 3.4.

3.6. $x\underbrace{y \dots y}_{n} = (xx)\underbrace{y \dots y}_{n} = x(x\underbrace{y \dots y}_{n})$ by 2.3 and 2.2.

3.7. It is enough to prove the statement for $l = 2$:

$$\begin{aligned} x\underbrace{y \dots y}_{n} &= x(\underbrace{y \dots y}_{n+1})\underbrace{y \dots y}_{n-1} && \text{by } 2.3 \\ &= x\underbrace{yy \dots y}_{n}\underbrace{y \dots y}_{n-1} && \text{by } 2.2. \end{aligned}$$

Proof of theorem 2.

$$\begin{aligned} (x \circ y) \circ z &= x\underbrace{y \dots y}_{n}\underbrace{z \dots z}_{n} \\ &= [(x\underbrace{y \dots y}_{n-1}z)(yz)]\underbrace{z \dots z}_{n-1} && \text{by } 3.2 \\ &= (xy\underbrace{z \dots z}_{n})\underbrace{(y\underbrace{z \dots z}_{n}) \dots (y\underbrace{z \dots z}_{n})}_{n-1} && \text{by } 3.2 \\ &= x\underbrace{(y\underbrace{z \dots z}_{n}) \dots (y\underbrace{z \dots z}_{n})}_{n} && \text{by } 2.2 \\ &= x \circ (y \circ z), \\ x \circ x &= x \quad \text{by } 2.3. \end{aligned}$$

Let m be the least multiple of n such that $m > k$. Then we get

$$\begin{aligned}(x\circ y)\circ z &= x\underset{n}{y\ldots y}\,\underset{n}{z\ldots z}\\ &= x\underset{n}{y\ldots y}\,\underset{m}{z\ldots z}\\ &= z(x\underset{n}{y\ldots y})\underset{m-k}{z\ldots z} && \text{by 2.1 and 2.3}\\ &= zx\underset{n}{y\ldots y}\,\underset{m-k}{z\ldots z} && \text{by 2.2,}\end{aligned}$$

$$\begin{aligned}(x\circ z)\circ y &= x\underset{n}{z\ldots z}\,\underset{n}{y\ldots y}\\ &= x\underset{m}{z\ldots z}\,\underset{n}{y\ldots y} && \text{by 3.7}\\ &= zx\underset{m-k}{z\ldots z}\,\underset{n}{y\ldots y} && \text{by 3.4}\\ &= (zx\underset{n}{y\ldots y})\underbrace{(z\underset{n}{y\ldots y})\ldots(z\underset{n}{y\ldots y})}_{m-k} && \text{by 3.2.}\end{aligned}$$

We now prove by induction that, for all natural numbers l,

$$(zx\underset{n}{y\ldots y})\underbrace{(z\underset{n}{y\ldots y})\ldots(z\underset{n}{y\ldots y})}_{l} = zx\underset{n}{y\ldots y}\,\underset{l}{z\ldots z}.$$

The case $l = 0$ is trivial. Now, we assume the equation for l and we get

$$(zx\underset{n}{y\ldots y})\underbrace{(z\underset{n}{y\ldots y})\ldots(z\underset{n}{y\ldots y})}_{l+1}$$

$$= (z\underset{n}{y\ldots y})\cdot$$

$$\cdot\underbrace{[(zx\underset{n}{y\ldots y})\underbrace{(z\underset{n}{y\ldots y})\ldots(z\underset{n}{y\ldots y})}_{l}]\ldots[(zx\underset{n}{y\ldots y})\underbrace{(z\underset{n}{y\ldots y})\ldots(z\underset{n}{y\ldots y})}_{l}]}_{k}$$

by 3.4

$$= (z\underset{n}{y\ldots y})[(zx\underset{n}{y\ldots y})\underbrace{(z\underset{n}{y\ldots y})\ldots(z\underset{n}{y\ldots y})}_{l}]\underbrace{(zx\underset{n}{y\ldots y}\,\underset{l}{z\ldots z})\ldots(zx\underset{n}{y\ldots y}\,\underset{l}{z\ldots z})}_{k-1}$$

by hypothesis.

For abbreviation, let

$$u = zx\underset{n}{y\ldots y}\,z\ldots z;$$

then we get

$$(z\underset{n}{xy\ldots y})\underbrace{(z\underset{n}{y\ldots y})\ldots(z\underset{n}{y\ldots y})}_{l+1}$$

$$= (z\underset{n}{y\ldots y})(zx\underset{n}{y\ldots y})\underbrace{(z\underset{n}{y\ldots y})\ldots(z\underset{n}{y\ldots y})}_{l}\underset{k-1}{u\ldots u} \qquad \text{by 3.3 and 2.3}$$

$$= \big(z(zx)\underset{n}{y\ldots y}\big)\underbrace{(z\underset{n}{y\ldots y})\ldots(z\underset{n}{y\ldots y})}_{l}\underset{k-1}{u\ldots u} \qquad \text{by 3.2}$$

$$= [z(zx\underset{n}{y\ldots y})]\underbrace{[z(z\underset{n}{y\ldots y})]\ldots[z(z\underset{n}{y\ldots y})]}_{l}\,\underset{k-1}{u\ldots u} \qquad \text{by 2.2 and 3.6}$$

$$= z[(zx\underset{n}{y\ldots y})\underbrace{(z\underset{n}{y\ldots y})\ldots(z\underset{n}{y\ldots y})}_{l}]\underset{k-1}{u\ldots u} \qquad \text{by 3.3}$$

$$= z\underset{k}{u\ldots u} \qquad \text{by assumption}$$

$$= zx\underset{n}{y\ldots y}\underset{l+1}{z\ldots z} \qquad \text{by 3.4 and the definition of } u.$$

Therefore, we get $(x\circ y)\circ z = (x\circ z)\circ y$.

$$(xy)\circ z = xy\underset{n}{z\ldots z}$$

$$= (x\underset{n}{z\ldots z})(y\underset{n}{z\ldots z}) \qquad \text{by 3.2}$$

$$= (x\circ z)(y\circ z),$$

$$x\circ(yz) = x\underset{n}{(yz)\ldots(yz)}$$

$$= (x\underset{n}{y\ldots y})(x\underset{n}{z\ldots z}) \qquad \text{by 3.3 and 3.1}$$

$$= (x\circ y)(x\circ z).$$

Therefore,

$$x\circ[(x\circ y)(x\circ z)] = x\circ\big(x\circ(yz)\big) = (x\circ x)\circ(yz) = x\circ(yz),$$

$$(xy)\circ x = xy\underset{n}{x\ldots x} = y\underset{k+n}{x\ldots x} = xy \qquad \text{by 3.4 and 3.5,}$$

$$(xy)\circ y = x\underset{n+1}{y\ldots y}$$

$$= (xy)(x\underset{n+1}{y\ldots y}) \qquad \text{by 3.6}$$

$$= (y\underset{n+k}{x\ldots x})(x\underset{n+1}{y\ldots y}) \qquad \text{by 3.5}$$

$$= (xy\underset{n}{x\ldots x})(x\underset{n+1}{y\ldots y}) \qquad \text{by 3.4}$$

$$= \underset{n+2}{(xy)\ldots(xy)} \qquad \text{by 3.1}$$

$$= xy \qquad \text{by 2.3,}$$

$$x\circ(xx) = x \qquad \text{by idempotency of } \cdot \text{ and } \circ.$$

Now, let $\mathfrak{G}_{k,n}^*$ be the class of groupoids $G^* = (G^*, \cdot)$ satisfying the equations

4.1. $x_0 x_1 \dots x_{k-1} x_k = x_k x_1 \dots x_{k-1} x_0$,

4.2. $x \underbrace{y \dots y}_{n} = x$,

4.3. $xx = x$.

It is easily seen that 4.2 implies 2.2. Therefore, we get, as an immediate consequence of theorems 1 and 2,

THEOREM 4. *Any groupoid in $\mathfrak{G}_{k,n}$ is the sum of a directed system of groupoids in $\mathfrak{G}_{k,n}^*$.*

The class $\mathfrak{G}_{k,n}^*$ was studied in [2], where the following result was proved:

THEOREM 5. *Let G^* be a groupoid of $\mathfrak{G}_{k,n}^*$. Then G^* is a full idempotent reduct of a module over the ring $\boldsymbol{R} = \boldsymbol{Z}[X]/(X^n - 1, X^k + X - 1)$.*

If r is the generator of $\boldsymbol{R}$ corresponding to X, then we can express the groupoid operation in terms of the module operations by $xy = rx + (1-r)y$. Conversely, we can express the module operations equationally by the groupoid operation. Therefore, we have

THEOREM 6. *Every groupoid in $\mathfrak{G}_{k,n}$ is a full idempotent reduct of a sum of a directed system of modules over the ring $\boldsymbol{R}$.*

Here, an algebra $\boldsymbol{B} = (B, G)$ is a full idempotent reduct of an algebra $\boldsymbol{A} = (A, F)$ iff $A = B$ and the algebraic operations of $\boldsymbol{B}$ are exactly the idempotent algebraic operations of $\boldsymbol{A}$.

Finally, we characterize the class of sums of directed systems of modules over the same ring with unit. As we mentioned above, the sums are defined if the algebras have no nullary operations. Therefore, we have to take a unary constant operation instead of the nullary one, that means, we consider a module over a ring $\boldsymbol{R}$ as an algebra $\boldsymbol{M} = \big(M, (+, -, c, (r)_{r \in \boldsymbol{R}})\big)$ of type $\big(2, 1, 1, (1)_{r \in \boldsymbol{R}}\big)$ satisfying

M1. $(x+y)+z = x+(y+z)$,

M2. $x+y = y+x$,

M3. $x+(-x) = c(x)$,

M4. $x+c(x) = x$,

M5. $c(x) = c(y)$,

M6. $1 \cdot x = x$,

M7. $(r+s)x = rx+sx$,

M8. $(r \cdot s)x = r(sx)$,

M9. $r(x+y) = rx+ry$,

where x, y, z denote elements of $\boldsymbol{M}$, and r, s elements of $\boldsymbol{R}$.

THEOREM 7. *An algebra* $\boldsymbol{N} = \big(N, (+, -, c, (r)_{r\in \boldsymbol{R}})\big)$ *of type* $\big(2, 1, 1, (1)_{r\in \boldsymbol{R}}\big)$, *where* $\boldsymbol{R}$ *is a ring with unit, is representable as a sum of a directed system of modules over this ring iff* $\boldsymbol{N}$ *satisfies* M1-M4, M6-M9 *and*

M5′. $(-1)x = -x$.

Proof. Let $\boldsymbol{N}$ satisfy these axioms. Then we get

$$\begin{aligned} c(-x) &= -x+-(-x) && \text{by M3}\\ &= -x+1x && \text{by M5}' \text{ and M8}\\ &= -x+x = c(x) && \text{by M6 and M3,}\end{aligned}$$

$$\begin{aligned} c(-x) &= (-1)(x+-x) && \text{by M5}', \text{ M3 and M9}\\ &= -c(x) && \text{by M3 and M9,}\end{aligned}$$

$$\begin{aligned} c(x)+c(x) &= c(x)+x+-x && \text{by M3}\\ &= x+-x = c(x) && \text{by M1, M2, M4 and M3,}\end{aligned}$$

$$\begin{aligned} c\big(c(x)\big) &= c(x)+-c(x) && \text{by M3}\\ &= c(x)+c(x) = c(x), \end{aligned}$$

$$\begin{aligned} c(x+y) &= x+y+-x+-y && \text{by M3, M5}' \text{ and M9}\\ &= c(x)+c(y) && \text{by M2 and M3,}\end{aligned}$$

$$\begin{aligned} 0x &= (1+-1)x = x+-x && \text{by M7 and M5}'\\ &= c(x) && \text{by M3,}\end{aligned}$$

$$\begin{aligned} c(rx) &= rx+-rx && \text{by M3}\\ &= 0x = c(x) && \text{by M5}', \text{ M8 and M7.}\end{aligned}$$

With these equations it is easily checked that the binary function $x\circ y = x+c(y)$ is a partition function of $\boldsymbol{N}$. If, on the other hand, $x+c(y) = x$ and M1-M4, M5′ and M6-M9 are satisfied, then $c(x)+c(y) = c(x)$ and $c(y) = y+-y = y+c(x)+-y = c(x)+c(y)$; therefore, $c(x) = c(y)$. Then theorem 1 completes the proof.

Sums of directed systems of groups are studied in [1], theorem 4.11. The result there is that sums of directed systems of groups are exactly the inverse semigroups which are a union of groups. In our special case we consider abelian groups and a ring with unit operating on them. As far as I know sums of directed systems of modules over the same ring are not characterized as previously known structures.

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