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A CONNECTION BETWEEN SPECTRAL RADIUS AND TRACE

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Wimmer [3] has shown that for a complex $(k \times k)$-matrix A

$$\overline{\lim_n} \sqrt[n]{|\mathrm{Tr} A^n|} = \varrho(A), \tag{1}$$

where $\mathrm{Tr} A$ is the usual trace of A and $\varrho(A)$ is the spectral radius of an operator on C^k corresponding to A.

It turns out that formula (1) remains true for much more complicated objects. Let M be a von Neumann algebra of operators on a Hilbert space H, let M^+ be a set of non-negative elements in M, and φ a trace defined in M^+. Let M_1 denote the ideal in M generated by the set of all T in M^+ with $\varphi(T) < \infty$. Then the trace φ has a unique extension (also denoted by φ) to a linear form on M_1. Observe that the left-hand and right-hand sides of equality (1) make sense for arbitrary $T \in M_1$. But not always we can get the equality. The aim of this paper is to show that if M is a *purely atomic* von Neumann algebra, e.g. if M is generated by its minimal projections, then (1) holds for every $T \in M_1$. We prove also that (1) characterizes these algebras.

The method presented in the proof is a generalization of Wimmer's idea. I am indebted to Dr. T. Pytlik for his help and suggestions.

Let M be a von Neumann algebra and let φ and M_1 be as above. The following proposition is well known ([2], I, § 6, Theorem 8):

PROPOSITION. *The inequality* $|\varphi(ST)| \leqslant \|S\| \varphi(|T|)$ *holds for every* $S \in M$ *and* $T \in M_1$.

COROLLARY 1. *We have*

$$\overline{\lim_n} \sqrt[n]{|\varphi(T^n)|} \leqslant \varrho(T) \quad \textit{for every } T \in M_1.$$

THEOREM. *Let M be a semifinite von Neumann algebra and let φ be a normal, faithful and semifinite trace in M. If M is purely atomic, then*

$$\varrho(T) = \overline{\lim_n} \sqrt[n]{|\varphi(T^n)|} \quad \textit{for every } T \in M_1. \tag{2}$$

If M is not purely atomic, then there exists a $T \in M_1$ such that $\varphi(T^n) = 0$ for $n = 1, 2, \ldots$, but $\varrho(T) \neq 0$.

So formula (2) gives a characterization of purely atomic von Neumann algebras among semifinite ones.

Proof. We may consider M as an algebra of operators on a Hilbert space H. If M is purely atomic, then there exists a maximal family E_i, $i \in I$, of mutually orthogonal projections in the center of the algebra M and $\sum_{i \in I} E_i$ is the identity on H. Each algebra ME_i is a factor of type 1, and so it is isomorphic to the full algebra $L(H_i)$ of operators on some Hilbert space H_i (cf. [2], p. 121). Let T_i denote the image of TE_i in $L(H_i)$. The trace $\varphi|_{ME_i}$ is proportional to the usual trace φ_i in $L(H_i)$ (cf. [2], I, § 6, Theorem 3, Corollary) and for $T \in M_1$ we have

$$\varphi(T) = \sum_{i \in I} \varphi(TE_i) = \sum_{i \in I} \sigma_i \varphi_i(T_i).$$

Let $T \in M_1$. By Corollary 1, only the inequality

$$\varrho(T) \leqslant \overline{\lim_n} \sqrt[n]{|\varphi(T^n)|}$$

must be shown. It is trivial when $\varrho(T) = 0$, and we may assume $\varrho(T) > 0$. Consider the function

$$f(z) = \sum_{n=1}^{\infty} \varphi(T^n) z^n$$

which is holomorphic on the circle with radius

$$R = \frac{1}{\overline{\lim_n} \sqrt[n]{|\varphi(T^n)|}}.$$

To prove that $R \leqslant 1/\varrho(T)$ we show that there exist singularities of the function $f(z)$ arbitrarily close to a circle of radius $1/\varrho(T)$. We have

$$f(z) = \sum_{n=1}^{\infty} \varphi(T^n) z^n = \sum_{n=1}^{\infty} \sum_{i \in I} \sigma_i \varphi_i(T_i^n) z^n = \sum_{n=1}^{\infty} \sum_{i \in I} \sigma_i \sum_k \lambda_{k,i}^n z^n,$$

where $\lambda_{k,i}$, $k = 1, 2, \ldots$, is the sequence (may be finite) of all non-zero eigenvalues of T_i.

The series on the right-hand side is absolutely summable for $|z| < R_0$, where

$$R_0^{-1} = \overline{\lim_n} \sqrt[n]{\sum_{i \in I} \sigma_i \sum_k |\lambda_{k,i}^n|} \leqslant \varrho(|T|) = \varrho(T),$$

so for $|z| < 1/\varrho(T)$ we may change the order of summation. Therefore

$$f(z) = \sum_{i\in I} \sigma_i \sum_k \sum_{n=1}^{\infty} \lambda_{k,i}^n z^n = \sum_{i\in I} \sigma_i \sum_k \frac{\lambda_{k,i} z}{1-\lambda_{k,i} z}.$$

Thus $1/\lambda_{k,i}$ are singularities of $f(z)$, and since

$$\varrho(T) = \sup_{i\in I} \varrho(T_i) = \sup_{i\in I} \sup_k |\lambda_{k,i}|,$$

we get the desired statement.

Now suppose M is not purely atomic. Then it is a direct product of a purely atomic von Neumann algebra and a non-trivial von Neumann algebra M_c without minimal projections.

Consider a restriction of φ to M_c. Since the trace φ is semifinite, in M_c there exists a projection P with a finite trace (we may assume $\varphi(P) = 1$), and since M_c has no minimal projection, there exists a sequence of projections $P_{k,l}$ for $k = 1, 2, \ldots$ and $l = 1, 2, \ldots, 2^k$ with the following properties:

1° $P_{k,1}, \ldots, P_{k,2^k}$ are mutually orthogonal for $k = 1, 2, \ldots$;

2° $P_{1,1}+P_{1,2} = P$ and $P_{k,2l-1}+P_{k,2l} = P_{k-1,l}$ for $k = 2, 3, \ldots$ and $l = 1, 2, \ldots, 2^{k-1}$;

3° $\varphi(P_{k,l}) = 1/2^k$ for $k = 1, 2, \ldots$ and $l = 1, 2, \ldots, 2^k$.

The sequence $P_{k,l}$ generates an abelian von Neumann algebra which is isomorphic to a von Neumann algebra $L^\infty(Z, \mu)$, where μ is a non-atomic probability measure on a compact set Z (cf. [2], p. 116-118). Since (Z, μ), as a measure space, is isomorphic to the interval $[0, 1]$ with Lebesgue measure, we have an isometric embedding of $L^\infty(0, 1)$ into M, which preserves the trace (on $L^\infty(0, 1)$ the trace is $\int_0^1$). If T in M_1 is the operator corresponding to the function $e^{2\pi i t}$, then $\varphi(T^n) = 0$ for $n = 1, 2, \ldots$, but $\varrho(T) \neq 0$.

Remarks. 1. Formula (2) remains true for all elements $T \in M$ such that $T^k \in M$ for an integer k.

2. Let G be a locally compact unimodular group and let $\mathrm{VN}(G)$ denote the von Neumann algebra of G. Every compact group has a purely atomic von Neumann algebra. But there exist non-compact (non-abelian) groups with that property. An example of such a group due to Fell has been given by Baggett ([1], p. 142).

Corollary 2. *Let G be a locally compact unimodular group. The formula*

$$\lim_n \sqrt[n]{\|f^{(n)}\|_2} = \overline{\lim_n} \sqrt[n]{|f^{(n)}(e)|},$$

where $f^{(n)} = f*f* \ldots *f$ (n times) and e is the identity in G, holds for every $f \in L^2(G) \cap \mathrm{VN}(G)$ if and only if $\mathrm{VN}(G)$ is purely atomic.

In particular,

COROLLARY 3. *For a compact group G and $f \in L^1(G) \cap L^2(G)$ we have*

$$\lim_n \sqrt[n]{\|f^{(n)}\|_1} = \lim_n \sqrt[n]{\|f^{(n)}\|_2} = \overline{\lim_n} \sqrt[n]{|f^{(n)}(e)|}.$$

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