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SOME REMARKS ON THE CONVERGENCE IN MEASURE AND ON A DOMINATED SEQUENCE OF OPERATORS MEASURABLE WITH RESPECT TO A SEMIFINITE VON NEUMANN ALGEBRA

BY

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0. Stinespring in [10] proved several dominated convergence theorems for operators measurable with respect to a semifinite von Neumann algebra. Padmanabhan [6], [7] gave some generalizations of these theorems. In this paper, we give a generalization of Theorems 5.3 and 5.4 of [7] (among other things, we do not assume the trace to be finite). Theorems 1 and 2 of [6] are obtained under weaker assumptions (Theorems 3.6 and 3.7). For we assume either m-local or weak m-local convergences in place of gross convergence. In the case where $\mathscr{A} = \mathscr{B}(\mathscr{H})$ is the von Neumann algebra of all bounded operators acting in a Hilbert space $\mathscr{H}$, while $m(p) = \dim p(\mathscr{H})$ is the ordinary (von Neumann) trace on $\mathscr{A}$, this means that either strong or weak convergence is assumed instead of convergence in operator norm. In Section 2 we investigate convergences in the *-algebra of measurable operators.

1. Basic definitions and notation. Throughout, $\mathscr{A}$ stands for a semifinite von Neumann algebra acting in a complex Hilbert space $\mathscr{H}$, and m is a faithful semifinite normal trace on $\mathscr{A}$. The centre of the algebra $\mathscr{A}$ is denoted by $\mathscr{Z}$. As is well known [3], there exist a locally compact Hausdorff space Ω, a Radon measure μ (unique up to equivalence of measures), a *-isomorphism $\Phi\colon \mathscr{Z} \to \mathscr{L}^\infty(\Omega, \mu)$, and a dimension function $d(\cdot)$ (unique up to multiplication by a positive real measurable function) mapping the projections in $\mathscr{A}$ to μ-measurable non-negative extended real-valued functions defined on Ω and satisfying conditions 1–9 of [9], Definition 1.4. In the sequel, we shall assume that the following equality holds (see [3]):

$$m(p) = \int_\Omega d(p)\,d\mu, \qquad p \in \mathrm{Proj}(\mathscr{A}),$$

where the set of all orthogonal projections in $\mathscr{A}$ is denoted by $\mathrm{Proj}(\mathscr{A})$. Let

$$|a| = (a^* a)^{1/2} = \int_0^\infty \lambda\, de_\lambda$$

be the spectral resolution. $\mathscr{L}_m(\mathscr{A}) = \mathscr{L}_m = \mathscr{L}$ stands for the *-algebra of operators measurable (m-measurable) in the sense of [4] (cf. [2]). For any $a \in \mathscr{L}$, let us put

$$\eta_a(\lambda) = m(e_\lambda^\perp), \quad \lambda > 0 \quad (m\text{-distribution of } a)$$

and

$$a(\alpha) = \inf\{0 \leqslant \lambda < \infty \colon m(e_\lambda^\perp) \leqslant \alpha\} \quad (\text{the rearrangement of } a).$$

We refer to [12] for the properties of the function $a(\alpha)$, $\alpha > 0$. Note that

$$\eta_a(\lambda) = \inf\{m(p^\perp)\colon \|ap\| \leqslant \lambda\}$$

and, in consequence,

$$\eta_{(a+b)}(\lambda+\delta) \leqslant \eta_a(\lambda) + \eta_b(\delta).$$

The subalgebra $\mathscr{S}_m(\mathscr{A}) = \mathscr{S}_m = \mathscr{S}$ of $\mathscr{L}$ is defined by

$$\mathscr{S} = \{a \in \mathscr{L}\colon a(\alpha) \to 0 \text{ as } \alpha \to \infty\} = \{a \in \mathscr{L}\colon m(e_\lambda^\perp) < \infty,\ \lambda > 0\}.$$

For $0 < \delta < \infty$, we define ([9], [5], [12])

$$\mathscr{L}_m^\sigma(\mathscr{A}) = \mathscr{L}_m^\sigma = \mathscr{L} = \{a \in \mathscr{L}\colon \|a\|_\sigma < \infty\},$$

where

$$\|a\|_\sigma = m(|a|^\sigma)^{1/\sigma} = \Big\{\int_0^\infty a^\sigma(\alpha)\, d\alpha\Big\}^{1/\sigma}.$$

For $\sigma \geqslant 1$, $\mathscr{L}^\sigma$ is a Banach space with norm $\|\cdot\|_\sigma$ ([9], [5], [12]). For $0 < \sigma < 1$, $\mathscr{L}^\sigma$ is a complete quasi-normed space (cf. [1] and [2]) with quasi-norm $\|\cdot\|_\sigma$.

$\bar{m}$ is a subadditive measure defined as follows (see [1] and [2]):

$$\bar{m}(p) = \begin{cases} 1 & \text{if } m(p) \geqslant 1, \\ m(p). & \text{if } m(p) < 1 \end{cases}$$

for $p \in \operatorname{Proj}(\mathscr{A})$.

2. Convergences in $\mathscr{L}_m(\mathscr{A})$.

Definition 2.1 ([4], [10]). A sequence of m-measurable operators $\{a_n\}$ is said to be *m-convergent* (*convergent in measure*) to a measurable operator a ($a_n \xrightarrow{m} a$) if one of the equivalent conditions (i)–(iii) is satisfied:

(i) $(a - a_n)(\alpha) \underset{n}{\to} 0$ for any $\alpha > 0$;

(ii) for any $\varepsilon > 0$,

$$m(e_\varepsilon^{(n)\perp}) \underset{n}{\to} 0,$$

where $|a - a_n| = \int_0^\infty \lambda de_\lambda^{(n)}$ is the spectral resolution of $|a - a_n|$;

(iii) for any $\varepsilon > 0$ there exists a sequence of projections from $\mathscr{A}$ such that

$$\|(a-a_n)p_n\| \leqslant \varepsilon, \quad m(p_n^{\perp}) \leqslant \varepsilon \quad \text{for } n \geqslant n_\varepsilon.$$

We refer to [10], [4] and [2] for the properties of m-convergence.

PROPOSITION 2.1. *If $a_n - b_n \xrightarrow{m} 0$ ($a_n, b_n \in \mathscr{L}$) and*

$$\eta_{a_n}(\lambda) \underset{n}{\rightarrow} \eta_a(\lambda) \quad (a_n(\lambda) \underset{n}{\rightarrow} a(\alpha))$$

at each point of continuity of the function $\eta_a(\lambda)$ $(a(\alpha))$, then

$$\eta_{b_n}(\lambda) \underset{n}{\rightarrow} \eta_a(\lambda) \quad (b_n(\alpha) \underset{n}{\rightarrow} a(\alpha))$$

at each point of continuity of the function $\eta_a(\lambda)$ $(a(\alpha))$.

Proof. Fix $\varepsilon > 0$. We have

$$a_n(\alpha+\varepsilon) \leqslant b_n(\alpha) + (a_n - b_n)(\varepsilon)$$

and

$$b_n(\alpha) \leqslant (b_n - a_n)(\varepsilon) + a_n(\alpha - \varepsilon)$$

for any $\alpha > \varepsilon$ and $n = 1, 2, \ldots$ Assume that for $n \geqslant n_\varepsilon$ we have $(a_n - b_n)(\varepsilon) < \varepsilon$. Then, for $n \geqslant n_\varepsilon$, the inequality

$$a_n(\alpha+\varepsilon) - \varepsilon \leqslant b_n(\alpha) \leqslant a_n(\alpha-\varepsilon) + \varepsilon$$

is true. If α is a point of continuity of the function $a(\alpha)$, then from the arbitrariness of $\varepsilon > 0$ the proposition follows.

For $\eta_a(\lambda)$ the proof is almost the same.

COROLLARY 2.1. *If $a_n \xrightarrow{m} a$, then $a_n(\alpha) \underset{n}{\rightarrow} a(\alpha)$ at each point of continuity of the function $a(\alpha)$ $(\eta_{a_n}(\lambda) \underset{n}{\rightarrow} \eta_a(\lambda))$.*

Using Corollary 2.1 we can give the following version of Fatou's lemma:

LEMMA 2.1. *If $a_n \xrightarrow{m} a$, $a, a_n \in \mathscr{L}_m(\mathscr{A})$, then*

$$\|a\|_\sigma \leqslant \liminf_n \|a_n\|_\sigma, \quad 0 < \sigma < \infty.$$

Proof. We have

$$\|a\|_\sigma^\sigma = \int_0^\infty a^\sigma(\alpha)\,d\alpha \leqslant \liminf_n \int_0^\infty a_n^\sigma(\alpha)\,d\alpha = \liminf_n \|a_n\|_\sigma^\sigma.$$

We next investigate how convergence in measure is related to convergence in measure of the spectral projections (cf. [7] and [8]).

THEOREM 2.1. *Let $a_n \xrightarrow{m} a$, $a, a_n \in \mathscr{L}$, $\eta_a(\lambda_0) < \infty$, where λ_0 is the continuity point of $\eta_a(\lambda)$. Then*

$$e_{\lambda_0}^{(n)\perp} \xrightarrow{m} e_{\lambda_0}^{\perp}, \quad \text{where } |a| = \int_0^\infty \lambda de_\lambda,\ |a_n| = \int_0^\infty \lambda de_\lambda^{(n)}.$$

Proof. We may suppose that $\lambda_k \downarrow \lambda_0$ and λ_0, λ_k are continuity points of $\eta_a(\lambda)$. By Corollary 2.1, for any k: $\eta_{a_n}(\lambda_0) \xrightarrow[n]{} \eta_a(\lambda_0)$ we have

$$\eta_{a_n}(\lambda_k) \xrightarrow[n]{} \eta_a(\lambda_k).$$

Hence

$$\eta_{a_n}(\lambda_0) - \eta_{a_n}(\lambda_k) < 3\varepsilon < 1 \quad \text{for } n \geqslant N_k$$

if $\eta_a(\lambda_0) - \eta_a(\lambda_k) < \varepsilon < \frac{1}{3}$ for $k \geqslant N_\varepsilon$. Now let $\psi(\lambda) = \chi_{(\lambda_0, \infty)}$ and

$$\varphi_k(\lambda) = \begin{cases} 1, & \lambda \geqslant \lambda_k, \\ (\lambda - \lambda_0)/(\lambda_k - \lambda_0), & \lambda_0 < \lambda < \lambda_k, \\ 0, & 0 \leqslant \lambda \leqslant \lambda_0. \end{cases}$$

Then $e_{\lambda_0}^{\perp} = \psi(|a|)$ and $e_{\lambda_0}^{(n)\perp} = \psi(|a_n|)$. Clearly, for any $\varepsilon > 0$, $k \geqslant N_\varepsilon$, and $n \geqslant N_k$,

$$\|\psi(|a|) - \psi(|a_n|)\|_{1,\bar{m}} \leqslant \|\psi(|a|) - \varphi_k(|a|)\|_{1,\bar{m}} + \|\varphi_k(|a|) - \varphi_k(|a_n|)\|_{1,\bar{m}} + \|\varphi_k(|a_n|) - \psi(|a_n|)\|_{1,\bar{m}}$$

and

$$\|\psi(|a|) - \varphi_k(|a|)\|_{1,\bar{m}} \leqslant \|\psi(|a|) - \varphi_k(|a|)\|\, \bar{m}(e_{\lambda_0}^{\perp} - e_{\lambda_k}^{\perp}) \leqslant m(e_{\lambda_0}^{\perp} - e_{\lambda_k}^{\perp}) = \eta_a(\lambda_0) - \eta_a(\lambda_k).$$

Similarly,

$$\|\psi(|a_n|) - \varphi_k(|a_n|)\|_{1,\bar{m}} \leqslant \eta_{a_n}(\lambda_0) - \eta_{a_n}(\lambda_k) < 3\varepsilon.$$

Moreover, $\varphi_k(|a_n|) \xrightarrow[n]{} \varphi_k(|a|)$ in $\mathscr{L}_{\bar{m}}^1(\mathscr{A})$, $k = 1, 2, 3, \ldots$ (see [2], Theorems 5.1, 4.1 and Corollary 5.1). As a consequence of the above chain of inequalities we obtain

$$\limsup_n \|\psi(|a|) - \psi(|a_n|)\|_{1,\bar{m}} \leqslant 2(\eta_a(\lambda_0) - \eta_a(\lambda_k)) < 2\varepsilon.$$

From the arbitrariness of ε and $k \geqslant N_\varepsilon$ it follows that

$$\psi(|a_n|) \xrightarrow[n]{} \psi(|a|) \text{ in } \mathscr{L}_{\bar{m}}^1, \quad \text{i.e.,} \quad e_{\lambda_0}^{(n)\perp} \xrightarrow{m} e_{\lambda_0}^{\perp}.$$

COROLLARY 2.2. *Let $a_n \xrightarrow{m} a$, $a, a_n \in \mathscr{S}_m(\mathscr{A})$. Then $e_\lambda^{(n)\perp} \xrightarrow{m} e_\lambda^{\perp}$ at each point of continuity of the function $\eta_a(\lambda)$.*

THEOREM 2.2. *Let* $a, a_n \in \mathcal{L}$ *and*

$$|a| = \int_0^\infty \lambda de_\lambda, \quad |a_n| = \int_0^\infty \lambda de_\lambda^{(n)}.$$

Assume that $e_\lambda^{(n)\perp} \xrightarrow{m} e_\lambda^\perp$, $\lambda > 0$, *at each point of continuity of the function* $\eta_a(\lambda)$. *Then* $|a_n| \xrightarrow{m} |a|$.

The theorem follows from Theorem 5.2 of [2] and the proof of Theorem 4.4 ("Converse") in [7].

Note that $e_\lambda^{(n)\perp} \xrightarrow{m} e_\lambda^\perp$ implies $e_\lambda^{(n)\perp} \xrightarrow{s} e_\lambda^\perp$ (see Proposition 2.3 and [10], Theorem 3.1).

Suppose that h is a positive normal functional on $\mathcal{A}$, i.e., $h(\cdot) = m(t\cdot)$, where $t \geqslant 0$, $\|t\|_1 = m(t) < \infty$. Let

$$\eta_a^h(\lambda) = h(e_\lambda^\perp), \quad \text{where } |a| = \int_0^\infty \lambda de_\lambda.$$

PROPOSITION 2.2. (i) *Let* $a_n \xrightarrow{m} a$, $a, a_n \in \mathcal{S}$. *Then* $\eta_{a_n}^h(\lambda) \underset{n}{\rightarrow} \eta_a^h(\lambda)$ *at each point of continuity of* $\eta_a^h(\eta)$.

(ii) *If* $\eta_{a_n}^h(\lambda) \rightarrow \eta_a^h(\lambda)$ *for any* h, *and* $\eta_{a_n}(\lambda) \rightarrow \eta_a(\lambda)$ *at each point of continuity of the functions* $\eta_a^h(\lambda)$ *and* $\eta_a(\lambda)$, *then* $|a_n| \xrightarrow{m} |a|$.

Proof. (i) By Theorem 2.1 we have $e_\lambda^{(n)\perp} \xrightarrow{m} e_\lambda^\perp$ at each point of continuity of $\eta_a(\lambda)$ and, in consequence,

$$t^{1/2} e_\lambda^{(n)\perp} t^{1/2} \xrightarrow{m} t^{1/2} e_\lambda^\perp t^{1/2}, \quad t^{1/2} e_\lambda^{(n)\perp} t^{1/2} \leqslant t,$$

and (see Theorem 3.1)

$$\eta_{a_n}^h(\lambda) = m(t^{1/2} e_\lambda^{(n)\perp} t^{1/2}) \underset{n}{\rightarrow} m(t^{1/2} e_\lambda^\perp t^{1/2}) = \eta_a^h(\lambda)$$

at each point of continuity of $\eta_a^h(\lambda)$.

(ii) The proof of (ii) is a slight modification of the proof of Lemma 2.2 ("Conversely") in [8].

DEFINITION 2.2 ([9], [5]). A sequence of measurable operators $\{a_n\}$ is said to *converge nearly everywhere* to a measurable operator a ($a_n \xrightarrow{\text{n.e.}} a$) if for any $\varepsilon > 0$ there exists a sequence of projections $\{p_n\}$ from $\mathcal{A}$ such that $\|(a - a_n) p_n\| \leqslant \varepsilon$, $p_n^\perp \downarrow 0$ and $p_n^\perp$ are finite for $n \geqslant n_\varepsilon$.

PROPOSITION 2.3 ([5], remark on p. 317). *Let* $a_n \xrightarrow{\text{n.e.}} a$ *and* $\|a_n\| \leqslant \varepsilon$, $n = 1, 2, \ldots$ *Then* $\|a\| \leqslant \varepsilon$ *and* $a_n \xi \rightarrow a\xi$ *for any* $\xi \in \mathcal{H}$.

DEFINITION 2.3 ([10], [11]). We say that a sequence of measurable operators $\{a_n\}$ *converges grossly* to a measurable operator a (in the terminology of [11] – *converges locally in measure*) if for any $\varepsilon > 0$ the sequence $d(e_\varepsilon^{(n)\perp})$ converges to zero in measure μ on each set $\mathcal{K} \subset \Omega$, $\mu(\mathcal{K}) < \infty$, where $|a - a_n| = \int_0^\infty \lambda de_\lambda^{(n)}$ is the spectral resolution of $|a - a_n|$.

DEFINITION 2.4. A sequence of measurable operators $\{a_n\}$ is said to *converge m-locally* to a measurable operator a ($a_n \xrightarrow{\text{m-l.}} a$) if, for any projection $p \in \operatorname{Proj}(\mathscr{A})$, $m(p) < \infty$, $a_n p \xrightarrow{m} ap$.

DEFINITION 2.5. A sequence of measurable operators $\{a_n\}$ is said to *converge weak m-locally* to a measurable operator a ($a_n \xrightarrow{\text{w.m-l.}} a$) if, for any projection $p \in \operatorname{Proj}(\mathscr{A})$, $m(p) < \infty$, $pa_n p \xrightarrow{m} pap$.

It is clear that $a_n \xrightarrow{\text{m-l.}} a$ implies $a_n \xrightarrow{\text{w.m-l.}} a$.

Remark 2.1. Let $\mathscr{A} = \mathscr{B}(\mathscr{H})$ and $m(p) = \dim p(\mathscr{H})$; then m-local (weak m-local) convergence coincides with strong (weak) convergence.

It is easy to check the following properties of m-local and weak m-local convergences to be used later.

PROPOSITION 2.4. *$a_n \xrightarrow{\text{w.m-l.}} a$ if and only if $qa_n p \xrightarrow{m} qap$ for any $p, q \in \operatorname{Proj}(\mathscr{A})$, $m(p)$, $m(q) < \infty$.*

PROPOSITION 2.5. *If $\{a_n\}$ is a sequence of self-adjoint measurable operators which converges weak m-locally to a measurable operator a, then a is also self-adjoint.*

PROPOSITION 2.6. *If $a_n \xrightarrow{\text{w.m-l.}} a$, $a_n \geqslant 0$, $n = 1, 2, \ldots$, then a is also non-negative.*

PROPOSITION 2.7. *Assume that $a_n \xrightarrow{\text{m-l.}} a$ ($a, a_n \in \mathscr{L}$) and $\|a_n p\| \leqslant \varepsilon$ for $n = 1, 2, \ldots$ and some $p \in \operatorname{Proj}(\mathscr{A})$. Then $\|ap\| \leqslant \varepsilon$.*

PROPOSITION 2.8. *Assume that $a_n \xrightarrow{\text{w.m-l.}} a$ and $\|pa_n p\| \leqslant \varepsilon$ for $n = 1, 2, \ldots$ and some $p \in \operatorname{Proj}(\mathscr{A})$. Then $\|pap\| \leqslant \varepsilon$.*

PROPOSITION 2.9. *Each of the convergences determined by Definitions 2.1–2.3 implies m-local convergence.*

Proof. Since $a_n \xrightarrow{m} a$ ($a_n \xrightarrow{\text{n.e.}} a$) implies gross convergence $a_n \to a$ ([10], Lemma 4.2), it suffices to prove that gross convergence implies m-local convergence. Let $a_n \to a$ grossly and

$$|a - a_n| = \int_0^\infty \lambda\, de_\lambda^{(n)}.$$

Besides, assume that $p \in \operatorname{Proj}(\mathscr{A})$, $m(p) < \infty$. In virtue of Definition 2.1 (ii) it is enough to show that, for any $\varepsilon > 0$,

$$m(f_\varepsilon^{(n)\perp}) \xrightarrow[n]{} 0, \quad \text{where } |(a - a_n)p| = \int_0^\infty \lambda\, df_\lambda^{(n)}.$$

It follows immediately from the equality

$$|(a - a_n)p| = (p|a - a_n|^2 p)^{1/2}$$

that $f_\varepsilon^{(n)\perp} \leqslant p$, which implies $d(f_\varepsilon^{(n)\perp}) \leqslant d(p)$. By the hypothesis,

$$m(p) = \int_\Omega d(p)\, d\mu < \infty$$

and $d(f_\varepsilon^{(n)\perp})$ tends to zero μ-locally. To complete the proof it is sufficient to make use of the Lebesgue dominated convergence theorem and of the relation

$$m(f_\varepsilon^{(n)\perp}) = \int_\Omega d(f_\varepsilon^{(n)\perp})\,d\mu.$$

In order to close this section, we give one more version of Fatou's lemma (cf. [10], Theorem 4.10, and [12], Theorem 2.9).

THEOREM 2.3. *Assume that* $a_n \xrightarrow{m\text{-}l} a$ $(a, a_n \in \mathscr{L}_m(\mathscr{A}))$. *Then*

$$\|a\|_\sigma \leqslant \liminf_n \|a_n\|_\sigma, \qquad 0 < \sigma < \infty.$$

Proof. For any $p \in \operatorname{Proj}(\mathscr{A})$, $m(p) < \infty$, by Lemma 2.1 we have

$$\|ap\|_\sigma \leqslant \liminf_n \|a_n p\|_\sigma \leqslant \liminf_n \|a_n\|_\sigma, \qquad 0 < \sigma < \infty.$$

It is therefore sufficient to prove that

$$\|a\|_\sigma = \sup\{\|ap\|_\sigma\colon\ p \in \operatorname{Proj}(\mathscr{A}),\ m(p) < \infty\}.$$

By [12], Proposition 2.4 (iii),

$$\|a\|_\sigma \geqslant \sup\{\|ap\|_\sigma\colon\ p \in \operatorname{Proj}(\mathscr{A}),\ m(p) < \infty\}.$$

Suppose that $a \in \mathscr{S}_m(\mathscr{A})$. Then for any $n = 1, 2, \dots$ we have

$$m(e_n - e_{1/n}) < \infty,$$

where

$$|a| = \int_0^\infty \lambda\, de_\lambda$$

and

$$\sup\{\|ap\|_\sigma\colon\ m(p) < \infty\} \geqslant \sup_n \|a(e_n - e_{1/n})\|_\sigma = \sup_n \big\||a|(e_n - e_{1/n})\big\|_\sigma$$
$$= \big\||a|\big\|_\sigma = \|a\|_\sigma.$$

Suppose next that $a \in \mathscr{L} \setminus \mathscr{S}$. It is sufficient to prove that

$$\sup\{\|ap\|_\sigma\colon\ p \in \operatorname{Proj}(\mathscr{A}),\ m(p) < \infty\} = \infty.$$

Let

$$|a| = \int_0^\infty \lambda\, de_\lambda.$$

Suppose that $\lambda_0 > 0$, $m(e_{\lambda_0}^\perp) = \infty$ and choose a projection q_n, $n^\sigma/\lambda_0^\sigma \leqslant m(q_n) < \infty$, $q_n \leqslant e_{\lambda_0}^\perp$, $n = 1, 2, \dots$ Then

$$q_n |a| q_n \geqslant \lambda_0 q_n.$$

Hence

$$\|q_n |a| q_n\|_\sigma \geqslant \lambda_0 m(q_n)^{1/\sigma} \geqslant n$$

and

$$n \leqslant \|q_n |a| q_n\|_\sigma \leqslant \big\||a| q_n\big\|_\sigma = \|aq_n\|_\sigma.$$

In other words,

$$\sup_n \|aq_n\|_\sigma = \infty,$$

which completes the proof of Theorem 2.3.

COROLLARY 2.3. *If* $a_n \xrightarrow{\text{w.m-l.}} a$, *then*

$$\|a\|_\sigma \leqslant \liminf_n \|a_n\|_\sigma.$$

Proof. Fix $p \in \operatorname{Proj}(\mathscr{A})$, $m(p) < \infty$. Then $pa_n \xrightarrow{\text{m-l.}} pa$. By Theorem 2.3,

$$\|pa\|_\sigma \leqslant \liminf_n \|pa_n\|_\sigma \leqslant \liminf_n \|a_n\|_\sigma.$$

On the other hand,

$$\begin{aligned} \|a\|_\sigma = \|a^*\|_\sigma &= \sup\{\|a^* p\|_\sigma \colon m(p) < \infty\} \\ &= \sup\{\|pa\|_\sigma \colon m(p) < \infty\} \leqslant \liminf_n \|a_n\|_\sigma. \end{aligned}$$

3. Several theorems on the convergence of a dominated sequence of measurable operators. We shall now give the following generalization of Theorems 5.3 and 5.4 in [7], ignoring, among other things, the finiteness of the trace. Let b_n be a sequence of non-negative operators belonging to $\mathscr{L}^1_m(\mathscr{A})$ and let

$$b_n \xrightarrow{m} b \in \mathscr{L}^1_m(\mathscr{A}), \qquad \|b_n\|_1 \to \|b\|_1.$$

THEOREM 3.1. *Let* $a_n \xrightarrow{m} a$ $(a, a_n \in \mathscr{L}_m(\mathscr{A}))$ *and let one of the conditions* (i), (ii) *be satisfied*:

(i) $|a_n|^\sigma \leqslant b_n$, $n = 1, 2, \ldots$, $0 < \sigma < \infty$,

(ii) $-b_n \leqslant (\operatorname{Re} a_n)^\sigma \leqslant b_n$, $-b_n \leqslant (\operatorname{Im} a_n)^\sigma \leqslant b_n$, $n = 1, 2, \ldots$,

under the assumption that the function $R^1 \ni \lambda \to \lambda^\sigma \in R^1$ *is defined. Then* $a \in \mathscr{L}^\sigma$ *and* $\|a - a_n\|_\sigma \to 0$.

Proof. (i) By Fatou's lemma,

$$\|a\|_\sigma \leqslant \liminf_n \|a_n\|_\sigma \leqslant \liminf_n \|b_n\|_1^{1/\sigma} = \|b\|_1^{1/\sigma} < \infty.$$

In other words, $a \in \mathscr{L}^\sigma$ and $\|a\|_\sigma^\sigma \leqslant \|b\|_1$. We shall now show that $a_n \to a$ in $\mathscr{L}^\sigma$. For this purpose we consider two cases.

Case 1. $\sigma \geqslant 1$. By assumption, $a_n \xrightarrow{m} a$, that is, $(a-a_n)(\alpha) \underset{n}{\to} 0$, and thus $(a-a_n)^\sigma(\alpha) \underset{n}{\to} 0$ for any $\alpha > 0$. Moreover,

$$(a-a_n)^\sigma(\alpha) \leqslant [a(\alpha/2)+a_n(\alpha/2)]^\sigma$$
$$\leqslant 2^{\sigma-1}[a^\sigma(\alpha/2)+a_n^\sigma(\alpha/2)] \leqslant 2^{\sigma-1}[a^\sigma(\alpha/2)+b_n(\alpha/2)].$$

Thus, by Proposition 2.1 and the Lebesgue dominated convergence theorem,

$$\|a-a_n\|_\sigma^\sigma = \int_0^\infty (a-a_n)^\sigma(\alpha)\,d\alpha \to 0.$$

Case 2. $0 < \sigma < 1$. The proof of case 2 differs only in the estimate

$$(a-a_n)^\sigma(\alpha) \leqslant [a(\alpha/2)+b_n(\alpha/2)]^\sigma \leqslant a^\sigma(\alpha/2)+a_n^\sigma(\alpha/2)$$
$$\leqslant a^\sigma(\alpha/2)+b_n^\sigma(\alpha/2).$$

(ii) Without loss of generality we may assume that $a_n = a_n^*$, $n = 1, 2, \ldots$ Let

$$a_n = \int_{-\infty}^{\infty} \lambda de_\lambda^{(n)}.$$

From condition (ii) we get at once

$$(a_n e_{\langle 0,\infty)}^{(n)})^\sigma = a_n^\sigma e_{\langle 0,\infty)}^{(n)} \leqslant e_{\langle 0,\infty)}^{(n)} b_n e_{\langle 0,\infty)}^{(n)}, \qquad n = 1, 2, \ldots,$$

and

$$-e_{(-\infty,0)}^{(n)} b_n e_{(-\infty,0)}^{(n)} \leqslant a_n^\sigma e_{(-\infty,0)}^{(n)} \leqslant e_{(-\infty,0)}^{(n)} b_n e_{(-\infty,0)}^{(n)}, \qquad n = 1, 2, \ldots$$

Thus

$$|a_n|^\sigma = a_n^\sigma e_{\langle 0,\infty)}^{(n)} \pm a_n^\sigma e_{(-\infty,0)}^{(n)} \leqslant e_{\langle 0,\infty)}^{(n)} b_n e_{\langle 0,\infty)}^{(n)} + e_{(-\infty,0)}^{(n)} b_n e_{(-\infty,0)}^{(n)}.$$

Hence

$$a_n^\sigma(\alpha) \leqslant 2b_n(\alpha/2)$$

and

$$(a-a_n)^\sigma(\alpha) \leqslant 2^{\sigma-1}[a^\sigma(\alpha/2)+2b_n(\alpha/4)] \quad \text{for } \sigma \geqslant 1.$$

For $0 < \sigma < 1$,

$$(a-a_n)^\sigma(\alpha) \leqslant a^\sigma(\alpha/2)+2b_n(\alpha/4).$$

Consequently, it suffices to use the Lebesgue dominated convergence theorem and Corollary 2.1 to prove that $a_n \to a$ in $\mathscr{L}^\sigma$.

COROLLARY 3.1. *If, under the assumptions of Theorem* 3.1 *omitting conditions* (i) *and* (ii),

(iii) $|a-a_n|^\sigma \leqslant b_n$, $n = 1, 2, \ldots$,

or

(iv) $|\mathrm{Re}\, a_n|^\sigma \leqslant b_n$, $|\mathrm{Im}\, a_n|^\sigma \leqslant b_n$, $n = 1, 2, \ldots$,

then $\|a-a_n\|_\sigma \to 0$.

Remark 3.1. In the proof of Theorem 3.1 we have only made use of

$$\|b_n\|_1 = \int_0^\infty b_n(\alpha)\,d\alpha \to \int_0^\infty b(\alpha)\,d\alpha = \|b\|_1 \quad \text{and} \quad b_n(\alpha) \to b(\alpha)$$

almost everywhere with respect to the Lebesgue measure on the half-line $(0, \infty)$. The following trivial example shows that these conditions can be satisfied despite the fact that m-convergence does not hold. Indeed, let $p, q \in \mathrm{Proj}(\mathscr{A})$, $p \sim q \neq 0$, $p \perp q$. Put $b_n = p$, $n = 1, 2, \ldots$, $b = q$. Of course, $b_n(\alpha) = b(\alpha)$, $\alpha > 0$, $\|b_n\|_1 = \|b\|_1$, $n = 1, 2, \ldots$ If $b_n \xrightarrow{m} b$, then

$$0 = qp = qb_n \xrightarrow{m} qb = q^2 = q,$$

which leads to a contradiction with the assumptions. In this example, $b_n \xrightarrow{m} p$. By putting $b_{2n} = p$, $b_{2n-1} = q$, $n = 1, 2, \ldots$, $b = q$, we obtain an example of the sequence $\{b_n\}$ which is not m-convergent and satisfies the required conditions.

We shall now give a few versions of Theorem 3.1, assuming the m-local convergence of the sequence $\{a_n\}$ (cf. [10], Theorems 4.6, 4.8 and 4.9). We assume that, for a sequence of operators $\{b_n\}$,

$$\|b_n\|_1 = \int_0^\infty b_n(\alpha)\,d\alpha \xrightarrow[n]{} \int_0^\infty b(\alpha)\,d\alpha = m(b) = \|b\|_1;$$

$b_n(\alpha) \to b(\alpha)$ almost everywhere with respect to the Lebesgue measure on the half-line $(0, \infty)$ $(0 \leqslant b,\ b_n \in \mathscr{L}^1_m(\mathscr{A}))$.

THEOREM 3.2. *Let us assume that a sequence of measurable operators* $\{a_n\}$ *converges m-locally to a measurable operator* a, $|a_n|^2 \leqslant c \in \mathscr{S}_m(\mathscr{A})$, $n = 1, 2, \ldots$, *and one of the conditions* (i), (ii) *is satisfied*:

(i) $|a_n|^\sigma \leqslant b_n$, $n = 1, 2, \ldots$, $0 < \sigma < \infty$,

(ii) $-b_n \leqslant (\mathrm{Re}\, a_n)^\sigma \leqslant b_n$, $-b_n \leqslant (\mathrm{Im}\, a_n)^\sigma \leqslant b_n$, $n = 1, 2, \ldots$, *under the assumption that the function* $R^1 \ni \lambda \to \lambda^\sigma \in R^1$ *is defined. Then* $a \in \mathscr{L}^\sigma$ *and* a_n *tends to* a *in* $\mathscr{L}^\sigma$.

The proof of this theorem is based on Theorem 3.1 and on the following

LEMMA 3.1. *Let* $a_n \xrightarrow{m\text{-}l} a$ $(a, a_n \in \mathscr{L})$ *and* $|a_n|^2 = a_n^* a \leqslant c \in \mathscr{S}$, $n = 1, 2, \ldots$ *Then* $a_n \xrightarrow{m} a$.

Proof. Fix $\varepsilon > 0$. Let

$$c = \int_0^\infty \lambda de_\lambda .$$

From the inequality $|a_n|^2 \leqslant c$ we get immediately

$$e_\varepsilon |a_n|^2 e_\varepsilon \leqslant ce_\varepsilon \leqslant \varepsilon 1.$$

Thus $\|a_n e_\varepsilon\|^2 = \|e_\varepsilon |a_n|^2 e_\varepsilon\| \leqslant \varepsilon$. Hence, by Proposition 2.7, $\|ae_\varepsilon\| \leqslant \varepsilon^{1/2}$, and so

$$\|(a-a_n)e_\varepsilon\| \leqslant 2\varepsilon^{1/2}.$$

What is more, $a_n = a_n e_\varepsilon + a_n e_\varepsilon^\perp$. By hypothesis, $a_n e_\varepsilon^\perp \xrightarrow{m} ae_\varepsilon^\perp$. In order to prove $a_n \xrightarrow{m} a$ it is sufficient to show that for any $\delta > 0$ there exists some n_δ and that, for $n \geqslant n_\delta$,

$$\|(a-a_n)p_n\| \leqslant \delta, \quad m(p_n^\perp) \leqslant \delta,$$

where $p_n \in \mathrm{Proj}(\mathscr{A})$, $n = 1, 2, \ldots$ Choose ε so that $\varepsilon \leqslant \delta^2/16$ and let

$$\|(a-a_n)e_\varepsilon^\perp p_n\| \leqslant \delta/2, \quad m(p_n^\perp) \leqslant \delta \quad \text{for } n \geqslant n_\delta.$$

Then, for $n \geqslant n_\delta$,

$$\|(a-a_n)p_n\| \leqslant \|(a-a_n)e_\varepsilon p_n\| + \|(a-a_n)e_\varepsilon^\perp p_n\| \leqslant 2\varepsilon^{1/2} + \delta/2 = \delta,$$

which completes the proof of the lemma.

Note that conditions (i) and (ii) in Theorem 3.2 may be replaced by one of the conditions

(iii) $|a-a_n|^\sigma \leqslant b_n$, $n = 1, 2, \ldots$;
(iv) $|\mathrm{Re}\, a_n|^\sigma \leqslant b_n$, $|\mathrm{Im}\, a_n|^\sigma \leqslant b_n$, $n = 1, 2, \ldots$

THEOREM 3.3. *Assume that*

$$a_n \xrightarrow{m\text{-}l.} a, \quad a_n^* \xrightarrow{m\text{-}l.} a^* \quad (a, a_n \in \mathscr{L}),$$

$$(|a_n|^2 + |a_n^*|^2)^{1/2} \leqslant c \in \mathscr{S}, \quad n = 1, 2, \ldots,$$

and one of the conditions (i)–(iv) *is satisfied*:

(i) $|a_n|^\sigma \leqslant b_n$, $n = 1, 2, \ldots$,
(ii) $|a-a_n|^\sigma \leqslant b_n$, $n = 1, 2, \ldots$,
(iii) $|\mathrm{Re}\, a_n|^\sigma \leqslant b_n$, $|\mathrm{Im}\, a_n|^\sigma \leqslant b_n$, $n = 1, 2, \ldots$,
(iv) $-b_n \leqslant (\mathrm{Re}\, a_n)^\sigma \leqslant b_n$, $-b_n \leqslant (\mathrm{Im}\, a_n)^\sigma \leqslant b_n$, $n = 1, 2, \ldots$,

under the assumption that the function $R^1 \ni \lambda \to \lambda^\sigma \in R^1$ is defined. Then $a \in \mathscr{L}^\sigma$ and $\|a-a_n\|_\sigma \to 0$, $0 < \sigma < \infty$.

The proof of the theorem will be preceded by a simple lemma.

LEMMA 3.2. *Let $a \in \mathscr{L}$, $a = a^*$ and $\|p|a|p\| \leqslant \varepsilon$ for some $p \in \mathrm{Proj}(\mathscr{A})$. Then $\|pap\| \leqslant 2\varepsilon$.*

Proof. Let

$$a = \int_{-\infty}^{\infty} \lambda de_\lambda$$

be the spectral resolution of a. Put

$$a^+ = ae_{\langle 0,\infty)} \quad \text{and} \quad a^- = ae_{(-\infty,0)}.$$

Obviously, $0 \leqslant a^+ \leqslant |a|$ and $0 \leqslant -a^- \leqslant |a|$. In consequence,

$$\|pa^+ p\| \leqslant \varepsilon \quad \text{and} \quad \|pa^- p\| \leqslant \varepsilon.$$

Hence we get at once

$$\|pap\| \leqslant \|pa^+ p\| + \|pa^- p\| \leqslant 2\varepsilon.$$

Proof of Theorem 3.3. Let

$$c = \int_0^{\infty} \lambda de_\lambda.$$

For any $\varepsilon > 0$, $n = 1, 2, \ldots$, we have

$$e_\varepsilon (|a_n|^2 + |a_n^*|^2)^{1/2} e_\varepsilon \leqslant ce_\varepsilon.$$

Hence and from the easily checked equality

$$(2|\mathrm{Re}\, a_n|^2 + 2|\mathrm{Im}\, a_n|^2)^{1/2} = (|a_n|^2 + |a_n^*|^2)^{1/2}$$

we obtain

$$2^{1/2} e_\varepsilon |\mathrm{Re}\, a_n| e_\varepsilon \leqslant ce_\varepsilon, \quad 2^{1/2} e_\varepsilon |\mathrm{Im}\, a_n| e_\varepsilon \leqslant ce_\varepsilon, \quad n = 1, 2, \ldots$$

Thus, by Lemma 3.2,

$$\|e_\varepsilon a_n e_\varepsilon\| \leqslant \|e_\varepsilon \mathrm{Re}(a_n) e_\varepsilon\| + \|e_\varepsilon \mathrm{Im}(a_n) e_\varepsilon\| \leqslant 2^{3/2} \varepsilon.$$

Using this inequality and the equality

$$a_n = e_\varepsilon a_n e_\varepsilon + e_\varepsilon^\perp a_n e_\varepsilon + e_\varepsilon^\perp a_n e_\varepsilon^\perp + e_\varepsilon a_n e_\varepsilon^\perp,$$

one can show, analogously as in the proof of Lemma 3.1 (see Proposition 2.8), that $a_n \xrightarrow{m} a$. To complete the proof it suffices to use Theorem 3.1 and Corollary 3.1.

Remark 3.2. It follows from Theorem 3.5 that $|\mathrm{Re}\, a_n| \leqslant c$, $|\mathrm{Im}\, a_n| \leqslant c$ may be assumed in place of

$$(|a_n|^2 + |a_n^*|^2)^{1/2} \leqslant c.$$

To finish with, let us note the verity of the following theorems. As to the sequence $\{b_n\}$, we now assume that

$$b_n \xrightarrow{\text{w.m-l.}} b, \quad \|b_n\|_1 \to \|b\|_1 \quad (0 \leqslant b,\ b_n \in \mathscr{L}_m^1(\mathscr{A})).$$

THEOREM 3.4. *If* $a_n \xrightarrow{m\text{-}l.} a$ $(a, a_n \in \mathscr{L})$ *and* $|a-a_n| \leqslant b_n$, $n = 1, 2, \ldots$, *then* $\|a-a_n\|_1 \to 0$.

COROLLARY 3.2. *As* $|a-a_n|^2 \leqslant b_n$, $n = 1, 2, \ldots$, *then* $\|a-a_n\|_2 \to 0$.

THEOREM 3.5. *If* $a_n \xrightarrow{w.m\text{-}l.} a$ $(a, a_n = a_n^* \in \mathscr{L})$ *and* $-b_n \leqslant a_n \leqslant b_n$, $n = 1, 2, \ldots$, *then* $a = a^* \in \mathscr{L}^1$ *and* $m(a_n) \to m(a)$.

COROLLARY 3.3. *Let* $a_n \xrightarrow{w.m\text{-}l.} a$, *and* $-b_n \leqslant \operatorname{Re} a_n \leqslant b_n$, $-b_n \leqslant \operatorname{Im} a_n \leqslant b_n$. *Then* $a \in \mathscr{L}^1$ *and* $m(a_n) \to m(a)$.

The proofs of these theorems are based on Fatou's lemma (Theorem 2.3, Corollary 2.3) and, for the case of gross convergence, included in [6] (Theorems 1 and 2, Corollaries 1.1 and 2.1).

Remark 3.3. Under the assumptions of Theorem 2 or Corollary 2.1 in [6] we have $\|a-a_n\|_{m_0,1} \to 0$ if the centre of $\mathscr{A}$ is countably decomposable (see [1] and [2], Theorem 4.1).

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