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MAPPING AN ARC ONTO A DENDRITIC CONTINUUM

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An *arc* is a Hausdorff continuum with only two non-cut points. A *dendritic continuum*, or *dendrite*, is a Hausdorff continuum such that each two of its points are separated by a third point. Thus neither an arc nor a dendrite is required to be separable. It is the purpose of this paper to prove that every dendrite is the image of an arc under a continuous map. This answers a question raised by Proizvolov [3].

Definition. If p is a point, M — a dendrite containing p, $\mathscr{S}$ — a collection of arcs such that $M = \bigcup \mathscr{S}$, each element of $\mathscr{S}$ has one end point at p, and $A \cap B = p$ for each two elements A and B of $\mathscr{S}$, then M is called a *simple dendrite growing from* p.

THEOREM 1. *If H is a proper subcontinuum of a dendrite K and $x \in H$ such that there is an arc in K with one end point at x and lying except for x in $K-H$, then there exists a maximal simple dendrite in K growing from x and lying except for x in $K-H$.*

Proof. Suppose xy is an arc lying except for x in $K-H$. Let $\mathscr{P}$ be the collection of all arcs with one end point at x and lying except for x in $K-H$. $\mathscr{P}$ is partially ordered by set inclusion. Let $\mathscr{S}$ be a maximal chain in $\mathscr{P}$, and let $A = \bigcup \mathscr{S}$. Suppose $\bar{A}$ has two non-cut points y and z distinct from x. But for each point w of A, $A - w$ contains a connected set with y and z as limit points, so that no point of K separates y from z. Hence $\bar{A}$ is an arc. Thus each such arc xy may be extended to a maximal arc. Now, again by use of the maximal principle, it may be shown that there exists a maximal collection $\mathscr{S}$ of maximal arcs each with one end point at x and lying except for x in $K-H$ and such that for each two elements A and B of $\mathscr{S}$, $A \cap B = x$. Let $L = \bigcup \mathscr{S}$. Suppose $y \in \bar{L}-L$. Some point w of K separates x from y. w belongs to at most one element A of $\mathscr{S}$. But $\bigcup \mathscr{S} - (A-x)$ is a connected set containing x and having y as a limit point. Therefore L is closed. It follows that L is the maximal simple dendrite desired.

THEOREM 2. *If M is a dendrite and p is a point of M, then there exist an arc X and a continuous map of X onto M that maps the end points of X onto p.*

Proof. The proof makes use of inverse limit systems over an initial interval of the class of ordinals. We first describe an inverse limit system $\{M_\alpha, f_\alpha^\beta\}$ of dendrites such that the inverse limit space M' is homeomorphic to M and then construct an inverse limit system $\{X_\alpha, g_\alpha^\beta\}$ of arcs such that the inverse limit space X is an arc and for each α there is a continuous map h_α of X_α onto M_α such that for $\alpha < \beta$, $f_\alpha^\beta \cdot h_\beta = h_\alpha \cdot g_\alpha^\beta$. We conclude that the map h, induced by the maps h_α, is a continuous map of X onto M'. $f \cdot h$ is then the desired map of X onto M.

1. Description of $\{M_\alpha, f_\alpha^\beta\}$. Let M_0 be a maximal simple dendrite in M growing from p. Suppose M_α has been defined and is a proper subcontinuum of M. For each point x of M_α such that there is an arc with one end point at x and lying except for x in $M - M_\alpha$, let $M_{x\alpha}$ be a maximal simple dendrite growing from x and lying except for x in $M - M_\alpha$.

Note that if $x \neq y$, then $M_{x\alpha} \cap M_{y\alpha} = \emptyset$.

Let $M_{\alpha+1}$ be the union of M_α and all such sets $M_{x\alpha}$. Clearly, $M_{\alpha+1}$ is connected. Suppose $z \in \bar{M}_{\alpha+1} - M_{\alpha+1}$. There exists a net $\{M_{x_n\alpha}\}$ of maximal simple dendrites lying except for x_n in $M - M_\alpha$ and such that for each n there is a point z_n in $M_{x_n\alpha}$ such that the net $\{z_n\}$ converges to z. Some subnet $\{x_{n_i}\}$ of the net $\{x_n\}$ converges to a point x of M_α. $\{M_{x_{n_i}\alpha}\}$ is then a net of disjoint connected subsets of M with both x and z in its limit interior, so that no point of M separates x from z. Therefore $M_{\alpha+1}$ is a subcontinuum of M.

Suppose γ is a limit ordinal and M_α has been defined for $\alpha < \gamma$ and is a proper subcontinuum of M. Then let

$$M_\gamma = \overline{\bigcup \{M_\alpha \mid \alpha < \gamma\}}.$$

Hence M_γ is a subcontinuum of M. It follows that, for each α, M_α is a dendrite and if $\alpha < \beta$, then $M_\alpha \subseteq M_\beta$. Now, for some λ, $M_\lambda = M$. We shall assume for the present that λ is a limit ordinal.

If $\alpha < \beta \leqslant \lambda$, let f_α^β be a map of M_β into M_α defined as follows. If $y \in M_\alpha$, let $f_\alpha^\beta(y) = y$. If $y \in M_\beta - M_\alpha$, then the component of $M_\beta - M_\alpha$, containing y, has only one boundary point x relative to M_β and $x \in M_\alpha$. In this case let $f_\alpha^\beta(y) = x$. In order to prove that f_α^β is continuous, suppose $L \subseteq M_\beta$ and y is a limit point of L. If $y \in M_\beta - M_\alpha$ and U is the component of $M_\beta - M_\alpha$ containing y, then $U \cap L \neq \emptyset$ and $f_\alpha^\beta(U \cap L) = f_\alpha^\beta(y)$, and hence $f_\alpha^\beta(y) \in f_\alpha^\beta(L)$. If $y \in M_\alpha$ and y is a limit point of $L \cap M_\alpha$, then $f_\alpha^\beta(y)$ is a limit point of $f_\alpha^\beta(L)$. Suppose $y \in M_\alpha$ and y is not a limit point of $L \cap M_\alpha$. If for some component U of $M_\beta - M_\alpha$, y is a limit point of $U \cap L$, then $f_\alpha^\beta(y) \in f_\alpha^\beta(L)$. Otherwise there is a net $\{U_n\}$ of components of $M_\beta - M_\alpha$ such that for each n there is a point y_n of $U_n \cap L$ such that the net $\{y_n\}$ converges to y. Some subnet of the net $\{f_\alpha^\beta(y_n)\}$ converges to a point x of M_α, and if $x \neq y$, then no point of M separates x from y. Hence $x = y$, so that $f_\alpha^\beta(y) \in \overline{f_\alpha^\beta(L)}$. Therefore f_α^β is continuous.

Now suppose $\alpha < \beta < \gamma$. Let $y \in M_\gamma - M_\beta$ such that $f_\beta^\gamma(y) \in M - M_\alpha$. Let U be the component of $M_\gamma - M_\beta$ containing y, and let V be the component of $M_\beta - M_\alpha$ containing $f_\beta^\gamma(y)$. Then $U \cup V$ is a subset of the component of $M_\alpha - M_\gamma$ containing y, so that $f_\alpha^\beta(f_\beta^\gamma(y)) = f_\alpha^\gamma(y)$. The last equation is clearly true in all other cases. It follows that $\{M_\alpha, f_\alpha^\beta\}$ is an inverse limit system such that, for $\alpha < \beta$, f_α^β is a contraction of M_β onto M_α. The inverse limit space M' is then a continuum.

In order to continue with the proof we need the following lemmas. All ordinals considered in the remainder of the proof are understood to be less than λ unless indicated otherwise.

LEMMA 1. *If* $\{x_\alpha, \alpha < \lambda\} \in M'$, U *is a component of* $M - M_\alpha$ *and, for some* $\beta > \alpha$, $x_\beta \in U$, *then* $x_\gamma \in U$ *for each* $\gamma > \alpha$.

Proof. Suppose $\gamma > \beta$. If $x_\gamma \in M_\beta$, then $f_\beta^\gamma(x_\gamma) = x_\gamma = x_\beta$. If $x_\gamma \in M - M_\beta$, then x_γ belongs to a component V of $M - M_\beta$ and hence $f_\beta^\gamma(x_\gamma) = \partial V = x_\beta$, so that $\overline{V} \subseteq U$. It follows that if $\gamma > \beta > \alpha$ and $x_\beta \in U$, then $x_\gamma \in U$.

We now show that if $\beta > \alpha$ and $x_\beta \in U$, then $x_{\alpha+1} \in U$. If $x_\beta \in M_{\alpha+1}$, then $f_{\alpha+1}^\beta(x_\beta) = x_{\alpha+1} = x_\beta$, so that $x_{\alpha+1} \in U$. Suppose $x_\beta \in M - M_{\alpha+1}$. Then the arc $x_\alpha x_\beta$ contains some point of $M_{x_\alpha\alpha} - x_\alpha$, since $M_{x_\alpha\alpha}$ is a maximal simple dendrite growing from x_α. If x is the last point of $x_\alpha x_\beta$ belonging to $M_{x_\alpha\alpha}$, then the arc xx_β lies except for x in the component of $M_\beta - M_{\alpha+1}$ containing x_β, and hence $f_{\alpha+1}^\beta(x_\beta) = x = x_{\alpha+1}$, so that again $x_{\alpha+1} \in U$. It follows that $x_\gamma \in U$ for each $\gamma > \alpha$.

LEMMA 2. *If the point* $\{x_\alpha, \alpha < \lambda\}$ *of* M' *has a cluster point* x *in* M_α, *then* $x_\beta = x$ *for each* $\beta \geqslant \alpha$.

Proof. Suppose, for some $\beta > \alpha$, that $x_\beta \in M - M_\alpha$. Let U be the component of $M - M_\alpha$ containing x_β. By Lemma 1, $x_{\alpha+1} \in U$. If, for some $\gamma > \alpha$, x_γ belongs to a component V of $M - M_{\alpha+1}$, then, again by Lemma 1, $x_\gamma \in V$ for each $\gamma > \alpha + 1$, so that $x \in \overline{V} \subseteq U$. If $x_\gamma \in M_{\alpha+1}$ for each $\gamma > \alpha$, then $f_{\alpha+1}^\gamma(x_\gamma) = x_\gamma = x_{\alpha+1}$, so that again $x \in U$. It follows that, for each $\beta > \alpha$, $x_\beta \in M_\alpha$ and hence $f_\alpha^\beta(x_\beta) = x_\beta = x_\alpha = x$.

LEMMA 3. *If* x *and* y *are two points of* M *such that* $x, y \in M - M_\alpha$ *for each* α, *then, for some* α, x *and* y *belong to different components of* $M - M_\alpha$.

Proof. Suppose, for each α, that x and y belong to the same component U_α of $M - M_\alpha$. Some point w of M separates x from y. Then $w \in U_\alpha$ for each α. But $\bigcup \{M_\alpha \mid \alpha < \lambda\}$ is a connected set having x and y as limit points, and hence $w \in M_\alpha$ for some α.

2. Proof that M' is homeomorphic to M. Since M is compact, each element of M' has a cluster point in M. Suppose that $\{x_\alpha, x < \lambda\}$ is an element of M' with two cluster points, x and y. If $x \in M_\alpha$ for some α, then it follows from Lemma 2 that $x_\beta = x$ for $\beta \geqslant \alpha$ and hence $x = y$. Similarly, if $y \in M_a$ for some a, then $x = y$. Therefore, $x, y \in M - M_a$ for each a and it follows from Lemma 1 that x and y belong to the same component of $M - M_a$. But this contradicts Lemma 3. Therefore each element of M' converges to a point of M. Let f be a map of M' into M defined by $f(\{x_a, a < \lambda\}) = \lim\{x_a, a < \lambda\}$. We shall prove that f is a homeomorphism of M' onto M.

In order to prove that f is one-to-one, suppose that each of $\{x_a, a < \lambda\}$ and $\{y_a, a < \lambda\}$ is an element of M' converging to the point x of M. If $x \in M_a$ for some a, then it follows from Lemma 2 that, for each $\beta \geqslant a$, $x_\beta = x$ and $y_\beta = x$, and hence $x_\beta = y_\beta$ for each β. If $x \in M - M_a$ for each a, then it follows from Lemma 1 that, for each $\beta > a$, x_β and y_β belong to the same component U of $M - M_a$, so that $f_a^\beta(x_\beta) = \partial U = x_a$ and $f_a^\beta(y_\beta) = \partial U = y_a$, and hence $x_a = y_a$ for each a.

We now show that f is onto. Let x be a point of M. If $x \in M_\beta$ for some β, then for each $\gamma \geqslant \beta$ let $x_\gamma = x$ and for each $a < \beta$ let $x_a = f_a^\beta(x_\beta)$. Suppose for each a, $x \in M - M_a$. Then for each a let U_a be the component of $M - M_a$ containing x and let $x_a = \partial U_a$. There is a point y such that $M - y$ is the union of two disjoint open sets V and W with $x_a \in V$ and $x \in W$. $y \in U_a$ and, for some $\gamma > a$, $y \in M_\gamma$. Therefore $M_a \subseteq V$ and $U_\gamma \subseteq W$, and hence $x_\gamma \in U_a$. It follows from Lemma 1 that $x_\beta \in U_a$ for $\beta > a$, and hence $f_a^\beta(x_\beta) = x_a$ for $\beta > a$. Therefore $\{x_a, a < \lambda\} \in M'$ and hence converges to some point y of M. For each a, $y \in \overline{U_{a+1}} \subseteq U_a$. Hence, for each a, x and y belong to the same component of $M - M_a$, and it follows from Lemma 3 that $x = y$.

It remains to prove that f is continuous. Suppose that D is a directed set, $\{x_{na}, a < \lambda\} \in M'$ for each n in D and that the net $\big\{\{x_{na}, a < \lambda\}, n \in D\big\}$ of points of M' converges to the point $\{y_a, a < \lambda\}$ of M'. For each n in D let

$$f(\{x_{na}, a < \lambda\}) = \lim\{x_{na}, a < \lambda\} = x_n$$

and

$$f(\{y_a, a < \lambda\}) = \lim\{y_a, a < \lambda\} = y.$$

Now, since M is compact, in order to prove that f is continuous it is sufficient to show that if x is a cluster point of $\{x_n, n \in D\}$, then $x = y$. So let x be a cluster point of $\{x_n, n \in D\}$. Some subnet $\{x_n, n \in E\}$ of $\{x_n, n \in D\}$ converges to x. The proof that $x = y$ involves the following three cases.

Case 1. For each a, $x \in M - M_a$. Let U be the component of $M - M_a$ containing x. There is an m in E such that if $n > m$, then $x_n \in U$. It then follows from Lemma 1 that $x_{n\beta} \in U$ for each $n > m$ and each $\beta > a$. Therefore $y_\beta \in U$ for each $\beta > a$, and hence $y \in \overline{U}$. Now, if $y = \partial U$, it follows from Lemma 2 that $y_\beta = \partial U$ for $\beta > a$. Therefore $y \in U$. Hence for each a both x and y belong to the same component of $M - M_a$. It follows from Lemma 3 that $x = y$.

Case 2. For some α, $x \in M_\alpha$, and for each α, $y \in M - M_\alpha$. Suppose there is an m in E such that $x_n \in M_\alpha$ for each $n > m$. It follows from Lemma 2 that $x_{n\beta} \in M_\alpha$ for each $n > m$ and each $\beta > \alpha$. But then $y_\beta \in M_\alpha$ for each $\beta > \alpha$, and hence $y \in M_\alpha$. Therefore there exists a subnet $\{x_n, n \in F\}$ of $\{x_n, n \in E\}$ of points of $M - M_\alpha$ converging to x. If for some component V of $M - M_\alpha$ there is an m in F such that $x_n \in V$ for $n > m$, then $x = \partial V = x_{n\alpha}$ for $n > m$, and hence $x = y_\alpha$. Otherwise there is a net $\{U_n, n \in G\}$ of distinct components of $M - M_\alpha$ such that $x_n \in U_n$ for each n in G, and $\{x_n, n \in G\}$ is a subnet of $\{x_n, n \in F\}$. Now $\{x_{n\alpha}, n \in G\}$ converges to y_α and, if $x \neq y_\alpha$, it may be shown that no point of M separates x from y_α. Therefore in either case $x = y_\alpha$. Thus we have shown that, for each α such that $x \in M_\alpha$, $x = y_\alpha$. Now if $x \in M_\alpha$, then $x \in M_\beta$ for $\beta > \alpha$. Therefore $x = y_\beta$ for $\beta > \alpha$, and hence $x = y$.

Case 3. For some α, $x, y \in M_\alpha$. If for each m in E there is an $n > m$ such that $x_{n\alpha} = x_n$, then $x = y_\alpha$, and since $y \in M_\alpha$, it follows from Lemma 2 that $y_\alpha = y$. Therefore there is an m in E such that $x_{n\alpha} \neq x_n$ for each $n > m$, so that, by Lemma 2, $x_n \in M - M_\alpha$. But then it may be shown as in case 2 that no point of M separates x from $y_\alpha = y$. It follows that $x = y$. This completes the proof that M' is homeomorphic to M.

3. Description of $\{X_\alpha, g_\alpha^\beta\}$. Suppose M is an arc from a to b. Let Y be the subspace $(M \times \{b\}) \cup (\{b\} \times M)$ of $M \times M$. Y is clearly an arc, and if g is a map of Y into M defined by $g((x, b)) = g((b, x)) = x$, then g is continuous. If X is an arc, h is a homeomorphism of X onto Y, and $f = g \cdot h$, then we say that *f folds X onto M.*

Now suppose M is a simple dendrite growing from p and γ is the cardinal number of components of $M - p$. Let the collection of closures of the components of $M - p$ be denoted by $\{M_\alpha \mid \alpha < \gamma\}$. Thus, for each α, M_α is an arc with one end point at p. Consider the set of ordinals less than γ with the order topology. For each $\alpha < \gamma$ there exist an arc Y_α from α to $\alpha + 1$ and a map g_α such that

(1) g_α folds Y_α onto M_α in such a way that $g_\alpha(\alpha) = g_\alpha(\alpha+1) = p$,

(2) if $\alpha + 1 < \beta < \gamma$, then $Y_\alpha \cap Y_\beta = \emptyset$, and

(3) if $\alpha + 1 < \gamma$, then $Y_\alpha \cap Y_{\alpha+1} = \alpha + 1$.

Let $Y = \bigcup \{Y_\alpha \mid \alpha < \gamma\} \cup \gamma$, and let g be a map of Y into M defined by $g(x) = g_\alpha(x)$ for $x \in Y_\alpha$ and $g(\gamma) = p$. Then Y is an arc from 0 to γ, and g is clearly continuous at each point x such that $g(x) \neq p$. Suppose $g(x) = p$. Let V be an open set containing p. Since M is compact and dendritic, there are at most finitely many α, say $\alpha_1, \ldots, \alpha_n$, such that M_α intersects $M - V$. Denote M_{α_i} by py_i. There is a point x_i of py_i between p and y_i such that $px_i \subseteq V$.

Let

$$U = Y - \bigcup \{g^{-1}(x_i y_i) \mid i = 1, \ldots, n\} = Y - \bigcup \{g_{\alpha_i}^{-1}(x_i y_i) \mid i = 1, \ldots, n\}.$$

Therefore U is open and $g(U) \subseteq V$, so that g is continuous at x. If X is an arc, h is a homeomorphism of X onto Y, and $f = g \cdot h$, then we say that f is a *simple map* of X onto M.

We now construct an arc X and a continuous map of X onto M. Let X_0 be an arc and h_0 a simple map of X_0 onto M_0. Suppose X_α and h_α have been defined in such a way that X_α is an arc and h_α is a continuous map of X_α onto M_α mapping the end points of X_α onto p. For each point q of M_α that is an end point of an arc lying except for q in $M - M_\alpha$ let x_q be a point of $h_\alpha^{-1}(q)$ and explode x_q into an arc X_q such that there is a simple map h_q of X_q onto the maximal simple dendrite $M_{q\alpha}$ growing from q and lying except for q in $M - M_\alpha$. Let $X_{\alpha+1}$ be the arc resulting from X_α by exploding all such points x_q, let $g_\alpha^{\alpha+1}$ be a map of $X_{\alpha+1}$ onto X_α such that if x is in some X_q, $g_\alpha^{\alpha+1}(x) = x_q$ and if x is not in any X_q, $g_\alpha^{\alpha+1}(x) = x$, and let $h_{\alpha+1}$ be a map of $X_{\alpha+1}$ onto $M_{\alpha+1}$ such that if x is in some X_q, $h_{\alpha+1}(x) = h_q(x)$ and if x is not in any X_q, $h_{\alpha+1}(x) = h_\alpha(x)$. Hence $f_\alpha^{\alpha+1} \cdot h_{\alpha+1} = h_\alpha \cdot g_\alpha^{\alpha+1}$. Clearly, $g_\alpha^{\alpha+1}$ and $h_{\alpha+1}$ are continuous. Now suppose γ is a limit ordinal, X_α and h_α have been defined for $\alpha < \gamma$, and g_α^β has been defined for $\alpha < \beta < \gamma$ in such a way that h_α is a continuous map of X_α onto M_α, $\{X_\alpha, g_\alpha^\beta\}$ is an inverse limit system of arcs and order preserving maps, and $f_\alpha^\beta \cdot h_\beta = h_\alpha \cdot g_\alpha^\beta$ for $\alpha < \beta < \gamma$. Now from what has been proved before there is a homeomorphism k_γ of M'_γ onto M_γ. We define X_γ as the inverse limit space of $\{X_\alpha, g_\alpha^\beta\}$, g_α^γ as the projection of X_γ on X_α, h'_γ as the map of X_γ onto M'_γ induced by $\{h_\alpha \mid \alpha < \gamma\}$, and h_γ as $k_\gamma \cdot h'_\gamma$. X_γ is then an arc and h_γ is a continuous map of X_γ onto M_γ mapping the end points of X_γ onto p. It follows by transfinite induction that there exist an arc X and a continuous map of X onto M that maps the end points of X onto p.

If x is a point of the dendrite M, then the *component number* of x, denoted by $c(x)$, is the cardinal number of the set of components of $M - x$. If H is a point set, let $|H|$ denote the cardinal number of H.

The construction used in the proof of Theorem 2 will also yield the following theorem, which is related to Theorem 1.4 of Chapter IX of [4]. The proof of Theorem 1.4 depends on certain metric considerations.

THEOREM 3. *If M is a dendrite, then there exist a simple closed curve X and a continuous light non-alternating map f of X onto M such that, for each x in M, $c(x) = |f^{-1}(x)|$.*

Finally, we note that the construction used in the proof of Theorem 2 leads to a new proof of a metrization theorem for dendrites, which is already known from the work of Eberhart [1] and Miller [2].

x is a *branch point* of M if $c(x) > 1$.

THEOREM 4. *If M is a dendrite such that* (1) *each collection of disjoint open sets in M is countable,* (2) *each arc in M is separable, and* (3) *the set of branch points of M is countable, then M is metrizable.*

Indication of proof. It follows from conditions (1) and (2) that each simple dendrite used in the construction is separable, and this together with (3) implies that each arc in the inverse limit system is separable. Condition (3) also implies that λ is countable. Hence the inverse limit space is homeomorphic to $[0, 1]$, and therefore M is metrizable.

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