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ON SETS OF CONFLUENT AND RELATED MAPPINGS IN THE SPACE Y^X

BY

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In this paper X and Y always mean compact Hausdorff spaces, and a continuum means a compact connected Hausdorff space.

Let Q be a subcontinuum of Y. A continuous mapping f from X onto Y is said to be

(i) *monotone* if the set $f^{-1}(Q)$ is connected (see [4], p. 123);

(ii) *confluent* if for each component C of $f^{-1}(Q)$ we have $f(C) = Q$ (see [1], p. 213);

(iii) *semi-confluent* if for each two components C_1 and C_2 of $f^{-1}(Q)$ we have either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$ (see [8], p. 252);

(iv) *weakly confluent* if there exists a component C of $f^{-1}(Q)$ such that $f(C) = Q$ (see [6], Sections 4 and 5);

(v) *pseudo-confluent* if for each pair of points $y, y' \in Q$ there exists a component C of $f^{-1}(Q)$ such that $y, y' \in f(C)$ (see [7], Corollary 1.4).

Let Φ denote the set of all mappings $f\colon X \to Y$ belonging to an arbitrary class among mentioned above. We consider a problem whether the set Φ is or is not closed in the space Y^X of all continuous mappings $f\colon X \to Y$ with the compact-open topology. This is a problem asked by Professor B. Knaster and Professor K. Sieklucki.

It is known that (see [5], p. 797)

PROPOSITION 1. *If Y is locally connected, then the set of all monotone onto mappings $f\colon X \to Y$ is closed in the space Y^X.*

We will prove such implication for confluent and semi-confluent mappings and we will prove that the conclusion of the theorem holds for pseudo-confluent mappings even without the assumption that Y is locally connected.

Firstly recall that the mapping $F\colon Y \to 2^X$ (here 2^X denotes the space of all closed subsets of X with the Vietoris topology) is called *upper semi-continuous* if the set $\{y\colon F(y) \subset G\}$ is open in Y whenever G is open in X (see [3], p. 173). We have (see, e.g., [4], p. 57, and [5], (γ), p. 798)

PROPOSITION 2. *If $f\colon X \to Y$ is continuous, then the mapping $f^{-1}\colon Y \to 2^X$ is upper semi-continuous.*

PROPOSITION 3. *If $H(f) = f^{-1}(B)$ for $B \in 2^Y$ constant, then the mapping $H\colon Y^X \to 2^X$ is upper semi-continuous.*

LEMMA 1. *If G is an open subset of X and B is a closed subset of Y, then the set $\Delta = \{g\colon g^{-1}(B) \subset G\}$ is open in Y^X.*

In fact, the mapping $H\colon Y^X \to 2^X$, defined by $H(g) = g^{-1}(B)$, is upper semi-continuous by Proposition 3. Thus, by the definition of upper semi-continuity, the set Δ is open in Y^X.

LEMMA 2. *If R is a closed subset of X and V is an open subset of Y, then the set $\Gamma = \{g\colon g(R) \subset V\}$ is open in Y^X.*

This follows immediately from the definition of the compact-open topology in Y^X.

LEMMA 3. *If $f\colon X \to Y$ is continuous and G is an open subset of X, then the set $U = \{y\colon f^{-1}(y) \subset G\}$ is open in Y.*

Indeed, the mapping $f^{-1}\colon Y \to 2^X$ is upper semi-continuous by Proposition 2. Therefore, by the definition of upper semi-continuity, the set U is open in Y.

Recall that, given three subsets A, B, C of a topological space, the set C is said to be *connected between A and B* provided $A, B \subset C$ and $C \neq M \cup N$, where $A \subset M$, $B \subset N$ and $\bar{M} \cap \bar{N} = \emptyset$. We have (see [4], §47, II, Theorem 3, p. 170)

PROPOSITION 4. *If A, B, C are compact sets and the set C is connected between A and B, then there exists a component K of C such that $A \cap K \neq \emptyset \neq B \cap K$.*

The proof of the following theorem partially coincides with the proof of Theorem of [5]:

THEOREM 1. *If a space Y is locally connected, then the set Φ of all confluent onto mappings $f\colon X \to Y$ is closed in the space Y^X.*

Proof. Let $f \in \bar{\Phi}$. It ought to be proved that f is confluent. Suppose that f is not confluent and let Q be a subcontinuum of Y, let C be a component of $f^{-1}(Q)$ and let $y_0 \in Q \setminus f(C)$. It follows from Proposition 4 that the set $f^{-1}(Q)$ is not connected between C and $f^{-1}(y_0)$. Thus, there are two closed sets A_1 and A_2 such that

(1.1) $$f^{-1}(Q) = A_1 \cup A_2, \quad A_1 \cap A_2 = \emptyset, \quad C \subset A_1 \quad \text{and} \quad f^{-1}(y_0) \subset A_2.$$

Since X is normal, there are open sets G_1 and G_2 such that

(1.2) $$A_1 \subset G_1, \quad A_2 \subset G_2 \quad \text{and} \quad G_1 \cap G_2 = \emptyset.$$

Since the mapping f is continuous and $G = G_1 \cup G_2$ is an open subset of X, the set

(1.3) $$U = \{y\colon f^{-1}(y) \subset G\}$$

is open by Lemma 3. Moreover, by (1.1) and (1.2), $Q \subset U$. The local connectedness of Y implies that there is a connected open set V such that

$$Q \subset V \quad \text{and} \quad \overline{V} \subset U. \tag{1.4}$$

We will show that

$$f^{-1}(\overline{V}) \subset G. \tag{1.5}$$

Indeed, if $x \in f^{-1}(\overline{V})$, then $f(x) \in \overline{V}$. Hence $f(x) \in U$ by (1.4). Since $x \in f^{-1}\big(f(x)\big)$, we conclude by (1.3) that $x \in f^{-1}\big(f(x)\big) \subset G$. Thus (1.5) holds.

Consider sets $\Delta \subset Y^X$ and $\Delta' \subset Y^X$ defined as follows:

$$\Delta = \{g\colon\, g^{-1}(\overline{V}) \subset G\}, \tag{1.6}$$

$$\Delta' = \{g\colon\, g^{-1}(y_0) \subset G_2\}. \tag{1.7}$$

Since G and G_2 are open and since $\overline{V}$ and $\{y_0\}$ are closed, we infer by Lemma 1 that sets Δ and Δ' are open in Y^X. Moreover, by (1.1), (1.2) and (1.5), we have

$$f \in \Delta \cap \Delta'. \tag{1.8}$$

Put

$$\Gamma = \big\{g\colon\, g\big(f^{-1}(Q)\big) \subset V\big\}. \tag{1.9}$$

Since $f^{-1}(Q)$ is closed and V is open, the set Γ is open in Y^X by Lemma 2. Moreover, $f \in \Gamma$, because $f\big(f^{-1}(Q)\big) = Q$ and $Q \subset V$ (by (1.4)). Consequently, the set $\Delta \cap \Delta' \cap \Gamma$ is open in Y^X which contains f (cf. (1.8)). Since $f \in \overline{\Phi}$, the set $\Phi \cap \Delta \cap \Delta' \cap \Gamma$ is non-empty. Thus there is a g such that

$$g \in \Phi \cap \Delta \cap \Delta' \cap \Gamma. \tag{1.10}$$

Since $g \in \Gamma$, we have, by (1.6), the decomposition

$$g^{-1}(\overline{V}) = \big(g^{-1}(\overline{V}) \cap G_1\big) \cup \big(g^{-1}(\overline{V}) \cap G_2\big) \tag{1.11}$$

of $g^{-1}(\overline{V})$ into two separated sets (cf. (1.2)). Since $g \in \Gamma$ (cf. (1.10)), we infer by (1.1) and (1.9) that $g(A_1) \subset \overline{V}$. Thus $A_1 \subset g^{-1}(\overline{V})$. This implies by (1.2) that $A_1 \subset g^{-1}(\overline{V}) \cap G_1$. Therefore the set $g^{-1}(\overline{V}) \cap G_1$ is non-empty. We conclude that there is a component K of the set $g^{-1}(\overline{V})$ which is contained in the set $g^{-1}(\overline{V}) \cap G_1$. It follows from $g \in \Delta'$ (cf. (1.10)) and (1.7) that $g^{-1}(y_0) \subset G_2$, thus $y_0 \notin g(K)$. But this contradicts the fact that g is confluent ($g \in \Phi$, cf. (1.10)), because $y_0 \in Q \subset \overline{V}$ (cf. (1.4)) and K is a component of $g^{-1}(\overline{V})$ such that $y_0 \notin g(K)$. The proof of Theorem 1 is complete.

THEOREM 2. *If a space Y is locally connected, then the set Φ of all semi-confluent onto mappings $f\colon X \to Y$ is closed in the space Y^X.*

Proof. Let $f \in \Phi$. We should prove that f is semi-confluent. Suppose that f is not semi-confluent, let Q be a subcontinuum of Y and let C_1 and C_2 be components of $f^{-1}(Q)$ such that the sets $f(C_1) \setminus f(C_2)$ and $f(C_2) \setminus f(C_1)$ are non-empty. Thus there are points y_1 and y_2 such that

$$(2.1) \qquad y_1 \in f(C_1) \setminus f(C_2) \quad \text{and} \quad y_2 \in f(C_2) \setminus f(C_1).$$

It follows from Proposition 4 that the set $f^{-1}(Q)$ is connected neither between C_1 and $f^{-1}(y_2)$ nor between C_2 and $f^{-1}(y_1)$. Thus there are closed sets A_1, A_2, B_1 and B_2 such that

$$(2.2) \qquad f^{-1}(Q) = A_1 \cup A_2, \quad A_1 \cap A_2 = \emptyset, \quad C_1 \subset A_1 \quad \text{and} \quad f^{-1}(y_2) \subset A_2,$$

and

$$(2.3) \qquad f^{-1}(Q) = B_1 \cup B_2, \quad B_1 \cap B_2 = \emptyset, \quad C_2 \subset B_1 \quad \text{and} \quad f^{-1}(y_1) \subset B_2.$$

Since X is normal, there are open sets G_1, G_2, H_1 and H_2 such that

$$(2.4) \qquad A_1 \subset G_1, \quad A_2 \subset G_2 \quad \text{and} \quad G_1 \cap G_2 = \emptyset$$

and

$$(2.5) \qquad B_1 \subset H_1, \quad B_2 \subset H_2 \quad \text{and} \quad H_1 \cap H_2 = \emptyset.$$

We can assume that $G_1 \cup G_2 = H_1 \cup H_2 = G$. Since f is continuous and G, G_2 and H_2 are open subsets of X, the sets

$$(2.6) \qquad U = \{y : f^{-1}(y) \subset G\},$$

$$(2.7) \qquad U_1 = \{y : f^{-1}(y) \subset H_2\},$$

and

$$(2.8) \qquad U_2 = \{y : f^{-1}(y) \subset G_2\}$$

are open by Lemma 3. Moreover, by (2.2)-(2.5), we have

$$(2.9) \qquad Q \subset U, \quad y_1 \in U_1 \quad \text{and} \quad y_2 \in U_2.$$

The local connectedness of Y implies that there is a connected open set V such that

$$(2.10) \qquad Q \subset V \quad \text{and} \quad \overline{V} \subset U.$$

We will show that

$$(2.11) \qquad f^{-1}(\overline{V}) \subset G.$$

Indeed, if $x \in f^{-1}(\overline{V})$, then $f(x) \in \overline{V}$. Hence $f(x) \in U$ by (2.10). Since $x \in f^{-1}(f(x))$, we conclude by (2.6) that $x \in f^{-1}(f(x)) \subset G$. Thus (2.11) holds.

Moreover, there are open sets V_1 and V_2 such that

$$(2.12) \qquad y_1 \in V_1, \quad y_2 \in V_2, \quad \overline{V}_1 \subset U_1, \quad \overline{V}_2 \subset U_2 \quad \text{and} \quad \overline{V}_1 \cap \overline{V}_2 = \emptyset.$$

We will show that

$$f^{-1}(\bar{V}_1) \subset H_2 \quad \text{and} \quad f^{-1}(\bar{V}_2) \subset G_2. \tag{2.13}$$

Indeed, if $x \in f^{-1}(\bar{V}_1)$, then $f(x) \in \bar{V}_1$. Hence $f(x) \in U_1$ by (2.12). Since $x \in f^{-1}(f(x))$, we conclude by (2.7) that $x \in f^{-1}(f(x)) \subset H_2$. The proof of the second inclusion is the same.

Consider sets $\Delta \subset Y^X$, $\Delta_1 \subset Y^X$ and $\Delta_2 \subset Y^X$ defined as follows:

$$\Delta = \{g\colon g^{-1}(\bar{V}) \subset G\}, \tag{2.14}$$

$$\Delta_1 = \{g\colon g^{-1}(\bar{V}_1) \subset H_2\}, \tag{2.15}$$

$$\Delta_2 = \{g\colon g^{-1}(\bar{V}_2) \subset G_2\}. \tag{2.16}$$

Since the sets G, H_2 and G_2 are open and since $\bar{V}$, $\bar{V}_1$ and $\bar{V}_2$ are closed, we infer by Lemma 1 that the sets Δ, Δ_1 and Δ_2 are open in Y^X. Moreover, by (2.11) and (2.13), we have $f \in \Delta \cap \Delta_1 \cap \Delta_2$. Thus

(2.17) $\Delta \cap \Delta_1 \cap \Delta_2$ is an open neighbourhood of f in Y^X.

Put

$$\Gamma = \{g\colon g(f^{-1}(Q)) \subset V\}, \tag{2.18}$$

$$\Gamma_1 = \{g\colon g(f^{-1}(y_1)) \subset V_1\} \tag{2.19}$$

and

$$\Gamma_2 = \{g\colon g(f^{-1}(y_2)) \subset V_2\}. \tag{2.20}$$

Since the sets $f^{-1}(Q)$, $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are closed and since the sets V, V_1 and V_2 are open, the sets Γ, Γ_1 and Γ_2 are open in Y^X by Lemma 2. Moreover, $f \in \Gamma \cap \Gamma_1 \cap \Gamma_2$, because $f(f^{-1}(Q)) = Q \subset V$, $f(f^{-1}(y_1)) = y_1 \in V_1$ and $f(f^{-1}(y_2)) = y_2 \in V_2$ by (2.10) and (2.12). Thus

(2.21) $\Gamma \cap \Gamma_1 \cap \Gamma_2$ is an open neighbourhood of f in Y^X.

It follows from (2.17) and (2.21) that the set $\Delta \cap \Delta_1 \cap \Delta_2 \cap \Gamma \cap \Gamma_1 \cap \Gamma_2$ is open in Y^X which contains f. Since $f \in \bar{\Phi}$, the set $\Phi \cap \Delta \cap \Delta_1 \cap \Delta_2 \cap \Gamma \cap \Gamma_1 \cap \Gamma_2$ is non-empty. Thus there is a g such that

$$g \in \Phi \cap \Delta \cap \Delta_1 \cap \Delta_2 \cap \Gamma \cap \Gamma_1 \cap \Gamma_2. \tag{2.22}$$

Since $g \in \Delta$, we have, by (2.14), two decompositions

$$g^{-1}(\bar{V}) = (g^{-1}(\bar{V}) \cap G_1) \cup (g^{-1}(\bar{V}) \cap G_2) \tag{2.23}$$

and

$$g^{-1}(\bar{V}) = (g^{-1}(\bar{V}) \cap H_1) \cup (g^{-1}(\bar{V}) \cap H_2) \tag{2.24}$$

of $g^{-1}(V)$ into two separated sets (cf. (2.4) and (2.5)).

Since the set $C_1 \cap f^{-1}(y_1)$ is non-empty (cf. (2.1)), there is a point $x_1 \in C_1$ such that $f(x_1) = y_1$. Since $g \in \Gamma$ (cf. (2.22)), we infer by (2.2) and (2.18) that $g(x_1) \in \overline{V}$. Thus $x_1 \in g^{-1}(\overline{V})$. This implies by (2.4) that $x_1 \in g^{-1}(\overline{V}) \cap G_1$. We conclude that there is a component K_1 of the set $g^{-1}(\overline{V})$ which is contained in the set $g^{-1}(\overline{V}) \cap G_1$ and $x_1 \in K_1$. It follows from $g \in \Delta_2$ (cf. (2.22)) and (2.16) that $g^{-1}(\overline{V}_2) \subset G_2$, thus $g(K_1) \cap \overline{V}_2 = \emptyset$. Moreover, since $g \in \Gamma_1$ (cf. (2.22)), $x_1 \in K_1$ and $f(x_1) = y_1$, we infer by (2.19) that $g(K_1) \cap V_1$ is non-empty. Thus

(2.25) $\quad K_1$ is a component of $g^{-1}(\overline{V})$, $g(K_1) \cap V_1 \neq \emptyset$ and $g(K_1) \cap \overline{V}_2 = \emptyset$.

Similarly, since the set $C_2 \cap f^{-1}(y_2)$ is non-empty (cf. (2.1)), there is a point $x_2 \in C_2$ such that $f(x_2) = y_2$. Since $g \in \Gamma$ (cf. (2.22)), we infer by (2.3) and (2.18) that $g(x_2) \in \overline{V}$. Thus $x_2 \in g^{-1}(\overline{V})$. This implies by (2.5) that $x_2 \in g^{-1}(\overline{V}) \cap H_1$. We conclude (cf. (2.24)) that there is a component K_2 of the set $g^{-1}(\overline{V})$ which is contained in the set $g^{-1}(\overline{V}) \cap H_1$ and $x_2 \in K_2$. It follows from $g \in \Delta_1$ (cf. (2.22)) and (2.15) that $g^{-1}(\overline{V}_1) \subset H_2$, thus $g(K_2) \cap \overline{V}_1 = \emptyset$. Moreover, since $g \in \Gamma_2$ (cf. (2.22)), $x_2 \in K_2$ and $f(x_2) = y_2$, we infer by (2.20) that $g(x_2) \in g(K_2) \cap V_2$; thus the set $g(K_2) \cap V_2$ is non-empty. Therefore, we have

(2.26) $\quad K_2$ is a component of $g^{-1}(\overline{V})$, $g(K_2) \cap \overline{V}_1 = \emptyset$ and $g(K_2) \cap V_2 \neq \emptyset$.

Consequently, K_1 and K_2 are components of the set $g^{-1}(\overline{V})$ and the sets $g(K_1) \setminus g(K_2)$ and $g(K_2) \setminus g(K_1)$ are non-empty by (2.12), (2.25) and (2.26). This means that the mapping g is not semi-confluent, because $\overline{V}$ is a subcontinuum of Y, which contradicts the fact that $g \in \Phi$ (cf. (2.22)). The proof of Theorem 2 is complete.

THEOREM 3. *The set Φ of all pseudo-confluent onto mappings $f\colon X \to Y$ is closed in the space Y^X.*

Proof. Let $f \in \overline{\Phi}$. We should prove that f is pseudo-confluent. Suppose that f is not pseudo-confluent. Let Q be a subcontinuum of Y and let y_1 and y_2 be points of Q such that there exists no component C of $f^{-1}(Q)$ with $y_1, y_2 \in f(C)$. This means by Proposition 4 that $f^{-1}(Q)$ is not connected between $f^{-1}(y_1)$ and $f^{-1}(y_2)$. Thus there are two closed sets A_1 and A_2 such that

(3.1) $\quad f^{-1}(Q) = A_1 \cup A_2, \quad A_1 \cap A_2 = \emptyset, \quad f^{-1}(y_1) \subset A_1 \quad$ and $\quad f^{-1}(y_2) \subset A_2$.

Since X is normal, there are open sets G_1 and G_2 such that

(3.2) $$A_1 \subset G_1, \quad A_2 \subset G_2 \quad \text{and} \quad G_1 \cap G_2 = \emptyset.$$

Put $G = G_1 \cup G_2$. Consider the sets $\Delta \subset Y^X$, $\Delta_1 \subset Y^X$ and $\Delta_2 \subset Y^X$ defined as follows:

(3.3) $$\Delta = \{g\colon g^{-1}(Q) \subset G\},$$

(3.4) $$\Delta_1 = \{g\colon g^{-1}(y_1) \subset G_1\},$$

(3.5) $$\Delta_2 = \{g\colon g^{-1}(y_2) \subset G_2\}.$$

Since the sets G, G_1 and G_2 are open and since the sets Q, $\{y_1\}$ and $\{y_2\}$ are closed, we infer by Lemma 1 that the sets Δ, Δ_1 and Δ_2 are open in Y^X. Moreover, by (3.1) and (3.2), we have $f \in \Delta \cap \Delta_1 \cap \Delta_2$. Thus

(3.6) $\quad \Delta \cap \Delta_1 \cap \Delta_2$ is an open neighbourhood of f in Y^X.

Since $f \in \overline{\Phi}$, the set $\Phi \cap \Delta \cap \Delta_1 \cap \Delta_2$ is non-empty by (3.6). Thus there is a g such that

$$g \in \Phi \cap \Delta \cap \Delta_1 \cap \Delta_2. \tag{3.7}$$

Since $g \in \Delta$ (cf. (3.7)), we have, by (3.3), the decomposition

$$g^{-1}(Q) = \big(g^{-1}(Q) \cap G_1\big) \cup \big(g^{-1}(Q) \cap G_2\big) \tag{3.8}$$

of $g^{-1}(Q)$ into two separated sets (cf. (3.2)). Thus any component of $g^{-1}(Q)$ is contained either in $g^{-1}(Q) \cap G_1$ or in $g^{-1}(Q) \cap G_2$. Let K be an arbitrary component of $g^{-1}(Q)$. If $K \subset g^{-1}(Q) \cap G_1$, then $y_2 \notin g(K)$, because $G_1 \cap G_2 = \emptyset$ and $g \in \Delta_2$ (cf. (3.2), (3.5) and (3.7)). If $K \subset g^{-1}(Q) \cap G_2$, then $y_1 \notin g(K)$, because $G_1 \cap G_2 = \emptyset$ and $g \in \Delta_1$ (cf. (3.2), (3.4) and (3.7)). Therefore, for each component K of $g^{-1}(Q)$ the set $g(K)$ fails to contain either y_1 or y_2. This means that g is not pseudo-confluent, because $y_1, y_2 \in Q$ and Q is a continuum. But this contradicts the fact that $g \in \Phi$ (cf. (3.7)). The proof of Theorem 3 is complete.

We have the following

PROBLEM. *Is the set of all weakly confluent onto mappings $f\colon X \to Y$ closed in the space Y^X?* (**P 986**)

The answer to this problem is positive if we assume additionally that X and Y are metric spaces. We will prove this.

Firstly we have (see [4], § 44, V, Theorem 2, p. 88)

PROPOSITION 5. *If Y is a metric space, then the topology of uniform convergence of Y^X coincides with the compact-open topology of Y^X.*

PROPOSITION 6. *In order that the sequence $\{f_n\}$ of continuous mappings of a metric space X onto a metric space Y be uniformly convergent to a mapping f it is necessary and sufficient that*

$$\lim_{n\to\infty} x_n = x$$

imply

$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

Recall that if $\{A_n\}$ is a sequence of subsets of a space X, then $\operatorname{Li}_{n\to\infty} A_n$ denotes the set of all points $x \in X$ for which every neighbourhood of x intersects A_n for almost all n and $\operatorname{Ls}_{n\to\infty} A_n$ denotes the set of all points $x \in X$ for which every neighbourhood of x intersects A_n for arbitrarily large n.

A sequence of subsets $\{A_n\}$ is said to *converge* to a set A (denoted by $\operatorname*{Lim}_{n\to\infty} A_n = A$) if $\operatorname*{Li}_{n\to\infty} A_n = A = \operatorname*{Ls}_{n\to\infty} A_n$. We have

COROLLARY 1. *If $\{f_n\}$ is a sequence of continuous mappings of a metric space X onto a metric space Y which is uniformly convergent to a mapping f and if Q is a closed subset of Y, then*

$$\operatorname*{Ls}_{n\to\infty} f_n^{-1}(Q) \subset f^{-1}(Q).$$

In fact, if $x \in \operatorname*{Ls}_{n\to\infty} f_n^{-1}(Q)$, then there is a sequence $\{x_n\}$ of points of X such that $\lim_{n\to\infty} x_n = x$ and $x_n \in f_n^{-1}(Q)$ for each $n = 1, 2, \ldots$ It follows from Proposition 6 that

$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

Since $x_n \in f_n^{-1}(Q)$, we infer that $f_n(x_n) \in Q$; thus $f(x) \in Q$, because Q is closed. But this means that $x \in f^{-1}(Q)$.

THEOREM 4. *If X and Y are metric spaces, then the set Φ of all weakly confluent onto mappings $f\colon X \to Y$ is closed in the space Y^X.*

Proof. Let $f \in \Phi$. It ought to be proved that f is weakly confluent. Let Q be an arbitrary subcontinuum of Y. It follows from Proposition 5 that there is a sequence $\{f_n\}$ of continuous mappings of X onto Y which is uniformly convergent to a mapping f, and $f_n \in \Phi$ for each $n = 1, 2, \ldots$ Since $f_n \in \Phi$, there is a component C_n of $f_n^{-1}(Q)$ such that $f_n(C_n) = Q$. We choose a convergent subsequence $\{C_{n_m}\}$ of the sequence $\{C_n\}$ (cf. [4], § 42, I, Theorem 1, p. 45, and § 42, II) and define $C = \operatorname*{Lim}_{m\to\infty} C_{n_m}$. It follows from Corollary 1 that

$$C = \operatorname*{Lim}_{m\to\infty} C_{n_m} \subset \operatorname*{Ls}_{n\to\infty} f_n^{-1}(Q) \subset f^{-1}(Q). \tag{4.1}$$

Moreover, the set C is a continuum (cf. [4], § 47, II, Theorem 4, p. 170). Take a component K of $f^{-1}(Q)$ such that

$$C \subset K. \tag{4.2}$$

We will show that

$$f(K) = Q. \tag{4.3}$$

Indeed, let $y \in Q$. Since $f_{n_m}(C_{n_m}) = Q$, there is a point $x_{n_m} \in C_{n_m}$ such that $f_{n_m}(x_{n_m}) = y$. We can assume that the sequence $\{x_{n_m}\}$ is convergent (X is a compact metric space) and we define

$$x = \lim_{m\to\infty} x_{n_m}.$$

It follows from Proposition 6 that

$$f(x) = \lim_{m\to\infty} f_{n_m}(x_{n_m}).$$

But $f_{n_m}(x_{n_m}) = y$ for each $m = 1, 2, \ldots$, thus $f(x) = y$. Moreover, by (4.1), $x \in C$. Therefore, there is a point $x \in K$ such that $f(x) = y$ (cf. (4.2)). Thus (4.3) holds.

Equality (4.3) implies that the mapping f is weakly confluent. The proof of Theorem 4 is complete.

The assumption in above theorems that mappings are onto is essential (cf. [5], Remarks, p. 799). This can be seen from the following

EXAMPLE 1. There is a sequence of homeomorphisms h_n of the interval $I = [0, 1]$ into the circle $S = \{(x, y): x^2+y^2 = 1\}$ which converges uniformly to a mapping h of I onto S which is not pseudo-confluent.

In fact, put

$$h_n(t) = \left(\cos 2\pi\left(\left(1-\frac{1}{n}\right)t+\frac{1}{n}\right), \sin 2\pi\left(\left(1-\frac{1}{n}\right)t+\frac{1}{n}\right)\right) \quad \text{for } t \in [0, 1].$$

The required conditions are easy to check.

The assumption in Theorems 1 and 2 that the space Y is locally connected is also essential. This can be seen from the following

EXAMPLE 2. There is a sequence of homeomorphisms h_n of the arcwise connected space M onto itself which converges uniformly to a mapping h_0 which is not semi-confluent (thus h is also not confluent; cf. [8], Proposition 2.1, p. 252).

Let C denote the Cantor ternary set lying in the unit interval $I = [0, 1]$ and let $a_n = 1/3^n$, $b_n = 2/3^n$ and $c_n = 2/3+1/3^{n+1}$ for $n = 1, 2, \ldots$ We define mappings $f_n: I \to I$ as follows:

$$a(x, y, x', y', t) = \frac{x'-y'}{x-y}(t-x)+x',$$

$$f_n(t) = \begin{cases} a(c_1, 1, a_1, 1, t) & \text{if } t \in [c_1, 1], \\ a(c_i, c_{i-1}, a_i, a_{i-1}, t) & \text{if } t \in [c_i, c_{i-1}] \text{ and } i = 2, \ldots, n, \\ a(b_1, c_n, b_{n+1}, a_n, t) & \text{if } t \in [b_1, c_n], \\ a(0, b_1, 0, b_{n+1}, t) & \text{if } t \in [0, b_1] \end{cases}$$

for $n = 1, 2, \ldots$

It is easy to check that

(a) f_n is a homeomorphism from I onto I,

(b) $f_n(C) = C$,

(c) the sequence $\{f_n\}$ converges uniformly to the mapping f_0, where

$$f_0(t) = \begin{cases} a(b_1, 1, 0, 1, t) & \text{if } t \in [b_1, 1], \\ 0 & \text{if } t \in [0, b_1]. \end{cases}$$

Put

$$N = (I \times \{0\}) \cup (C \times I),$$

$$g_n(x, y) = (f_n(x), y) \quad \text{for each } (x, y) \in N \text{ and } n = 0, 1, 2, \ldots$$

It follows from (b) that g_n are well defined. Moreover, by (a), g_n are homeomorphisms for $n = 1, 2, \ldots$, and, by (c), $\{g_n\}$ converges uniformly to g_0. However, the mapping f_0 is monotone and g_0 is confluent. We waste the confluence of g_0 in the following way.

We define an equivalence relation ϱ on N as follows:

$(x, y)\,\varrho(x', y')$ if and only if either $(x, y) = (x', y')$ or $x = x' = 0$, $y, y' \in [1/4, 3/4]$ and $|1/2 - y| = |1/2 - y'|$.

Denote by φ the canonical mapping from N onto N/ϱ. Put $M = N/\varrho$ and $h_n(q) = \varphi\big(g_n(\varphi^{-1}(q))\big)$ for each $q \in N/\varrho$ and $n = 0, 1, 2, \ldots$

It is obvious that M and h_n satisfy the required conditions.

The set of all confluent onto mappings $f\colon X \to Y$ and the set of all semi-confluent onto mappings $f\colon X \to Y$ are not closed in Y^X by any chance. This fact is more general.

Let $\mathscr{D}$ be an arbitrary class of continuous mappings, which contains a class of all homeomorphisms and is such that if $f \in \mathscr{D}$, then hf and fh belong to $\mathscr{D}$ whenever h is a homeomorphism.

We say that $\mathscr{D}$ has *property* (p) if the conditions $f\colon X \to Y$ and $f \in \mathscr{D}$ imply that for each component Q of Y and each component K of $f^{-1}(Q)$ there is $f|K \in \mathscr{D}$.

In particular, the class of monotone mappings, the class of confluent mappings and the class of semi-confluent mappings have property (p) (see [1], I, p. 213; [8], Theorem 3.7, p. 255).

THEOREM 5. *If a class $\mathscr{D}$ has property* (p) *and for each two compact metric spaces X and Y the set of all onto mappings $f\colon X \to Y$ belonging to the class $\mathscr{D}$ is closed in Y^X, then each continuous mapping of X onto Y belongs to $\mathscr{D}$.*

Proof. We use the idea of the construction of Example 2. Namely, let r be a continuous mapping from X onto Y and let C and f_n for $n = 0, 1, 2, \ldots$ be such as in Example 2.

Put $N = C \times X$ and $g_n(t, x) = \big(f_n(t), x\big)$ for each $(t, x) \in N$ and $n = 0, 1, 2, \ldots$

We have

(5.1) g_n is a homeomorphism for $n = 1, 2, \ldots$,

(5.2) g_n converges uniformly to g_0.

We define an equivalence relation ϱ on N as follows:

$(t, x)\,\varrho(t', x')$ if and only if either $(t, x) = (t', x')$ or $t = t' = 0$ and $r(x) = r(x')$.

Denote by φ the canonical mapping from N onto N/ϱ. Put $M = N/\varrho$ and $h_n(q) = \varphi\big(g_n(\varphi^{-1}(q)\big)$ for each $q \in N/\varrho$ and $n = 0, 1, 2, \ldots$ It follows from (5.1) that

(5.3) h_n is a homeomorphism for $n = 1, 2, \ldots$

Moreover, by (5.2),

(5.4) $\{h_n\}$ converges uniformly to h_0.

Since the set of all onto mappings $f: M \to M$ belonging to the class $\mathscr{D}$ is closed in M^M and $h_n \in \mathscr{D}$ (by (5.3)), we infer that $h_0 \in \mathscr{D}$ (by (5.4)). Since $\varphi(\{0\} \times X)$ is a component of M and $\varphi(\{2/3\} \times X)$ is a component of $h_0^{-1}(\varphi(\{0\} \times X))$, we conclude that $h = h_0|\varphi(\{2/3\} \times X)$ is a mapping from $\varphi(\{2/3\} \times X)$ onto $\varphi(\{0\} \times X)$ and $h \in \mathscr{D}$, because $\mathscr{D}$ has property (p). It is obvious that $\varphi(\{2/3\} \times X) = X$, $\varphi(\{0\} \times X) = Y$ and $h = r$ (the equalities are given with respect to homeomorphisms). Thus $r \in \mathscr{D}$.

COROLLARY 2. *If a class $\mathscr{D}$ has property* (p) *and for each two compact metric spaces X and Y the set of all onto mappings $f: X \to Y$ belonging to the class $\mathscr{D}$ is closed in Y^X, then $\mathscr{D}$ coincides with the class of all continuous mappings onto compact metric spaces.*

One can see from the proof of Theorem 5 that

COROLLARY 3. *If r is a continuous mapping from a compact metric space X onto a metric space Y, then there is a compact metric space M such that X and Y are subspaces of M and there is a sequence of homeomorphisms of M onto M which is uniformly convergent to a mapping h such that $h|X = r$.*

REMARKS. Recall that a continuous mapping f from X onto Y is said to be *open* if f maps every open set in X onto an open set in Y (see [10], p. 348). The set of all open onto mappings $f: X \to Y$ is not closed in Y^X, even if $X = Y = I = [0, 1]$. Moreover, there is a sequence of homeomorphisms of I onto I such that its limit is a monotone mapping which is not open (for example the sequence $\{f_n\}$ defined in Example 2). Since any confluent mapping onto a locally connected metric space is a composition of two mappings, one of which is monotone and the other is open (see [6], Corollary 5.2, p. 109), one can conjecture that any confluent mapping onto a locally connected metric space is a limit of a sequence of open mappings. But this is not in general true. This can be seen from the following

EXAMPLE 3. There are locally connected metric continua M and N such that there is no open mapping from M onto N, but there is a monotone mapping from M onto N.

Put $A_i = \{(i, y): -1 \leqslant y \leqslant 1\}$ for $i = 0, 1$ and $M = A_0 \cup A_1 \cup I$. Let a mapping $g: M \to g(M) = N$ be such that g maps I onto a point and $g|A_i$ for $i = 0, 1$ is homeomorphism. It is obvious that g is monotone.

Suppose, on the contrary, that there is an open mapping f from M onto N. Then the set $f^{-1}(t)$ is 0-dimensional for each $y \in Y$, and thus f is light (for the definition see [12], p. 130). Therefore, by Theorem (4.1) of [9], the continuum M must contain a point of order larger than 3 (for the definition, see [4], § 51, I, p. 274), a contradiction.

If Y is a locally connected metric space, then the set of all locally confluent (see [2], p. 239) onto mappings $f: X \to Y$ is closed in Y^X by Theorem 1, because it is equal to the set of all confluent mappings (see [6], Corollary 5.2). Similarly, if X is a locally connected metric space, then the set of all quasi-monotone (see [12], p. 294) onto mappings $f: X \to Y$ is closed in Y^X by Theorem 1, because it is equal to the set of all confluent mappings (see [11], Theorem (2.1), p. 137, and Theorem (2.3), p. 138; see also [1], p. 214, and IX, p. 215). But the set of all locally weakly confluent (see [9]) onto mappings is not closed in Y^X even if X is equal to I and Y is a unit circle. It is easy to see this.

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