COLLOQUIUM MATHEMATICUM

VOL. XV 1966 FASC. 2

THE LIMITS OF INDETERMINATION FOR RIEMANN SUMMATION IN TERMS OF BESSEL FUNCTIONS

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1. Introduction. The Riemann summation method $(R, 2k), k = 1, 2, \ldots$, assigns to a series $u_0+u_1+\ldots+u_n+\ldots$ the value

$$\lim_{a\downarrow 0}\left[u_0+\sum_{n=1}^{\infty}u_n\left(\frac{\sin na}{na}\right)^{2k}\right], \tag{1}$$

if this limit exists.

Let $s_0, s_1, \ldots, s_n, \ldots$ be the sequence of the partial sums of the given series $\sum u_n$, and denote by $\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots$ the sequence of the partial sums of its $(R, 2k)$ transform as indicated by (1). Assuming henceforth $\{s_n\}$ to be bounded, define

$$s^* = \limsup s_n, \quad s_* = \liminf s_n, \quad \sigma^* = \limsup \sigma_n, \quad \sigma_* = \liminf \sigma_n.$$

Zygmund [11] has shown that the smallest value of λ, say λ_{2k}, which satisfies

$$\tfrac{1}{2}(s^*+s_*)-\tfrac{1}{2}\lambda(s^*-s_*) \leqslant \sigma_* \leqslant \sigma^* \leqslant \tfrac{1}{2}(s^*+s_*)+\tfrac{1}{2}\lambda(s^*-s_*) \tag{2}$$

for all bounded sequences $\{s_n\}$ and their $(R, 2k)$ transforms is $\lambda_2 = \frac{1}{2}(e^2-5)$ when $k=1$. He remarked (loc. cit.) that his method can be used to determine λ_{2k} for $k = 2, 3, \ldots$ as well.

His proof consists of representing λ_2 as the total variation of the function $x^{-2}\sin^2 x$, $0 < x < \infty$, and then computing this value by contour integration.

Here this problem is re-examined, λ_{2k} determined explicitly in terms of Bessel functions (formula (8)) and a generating function obtained (formulas (9) and (10)). Some numerical discussion is included (§ 4).

2. The computation. Following Zygmund, we note that

$$\lambda_{2k} = 1+2\sum_{r=1}^{\infty}(1+a_r^2)^{-k}, \quad k = 1, 2, \ldots, \tag{3}$$

where $0 < a_1 < a_2 < \ldots$, yield the successive maxima of $x^{-2k}\sin^{2k}x$, $0 < x < \infty$. As he points out, a_r is the r-th positive zero of the function $\sin x - x\cos x$.

At this point, we depart from Zygmund's line of argument and evaluate (3) by means of Bessel functions, instead of employing contour integration. This can be done because a_r is obviously the r-th positive zero of $J_{3/2}(x)$, the Bessel function of the first kind and order $3/2$. This function is well-known to have infinitely many zeros, all of them real ([10], p. 482).

(Indeed, the reality of the zeros of $J_\nu(x)$ is essentially an immediate consequence of the reality of the eigenvalues, μ^2, of the Sturm-Liouville system

$$y'' + \{\mu^2 - x^{-2}(\nu^2 - \tfrac{1}{4})\}y = 0, \qquad y(0) = y(1) = 0,$$

when $\nu > -1/2$. When $-1/2 \geqslant \nu > -1$, the same reasoning applies to the system in which the condition $y(0) = 0$ is replaced in an obvious fashion by a slightly more complicated one satisfied by $x^{1/2}J_\nu(x)$ for such ν.)

Knowing the zeros of $J_{3/2}(x)$ to be real, so that $\{a_r\}$ is the set of its positive zeros, we rewrite (3):

$$\begin{aligned}
\lambda_{2k} - 1 &= 2\sum_{r=1}^{\infty} a_r^{-2k}(1 + a_r^{-2})^{-k} \\
&= \frac{2}{(k-1)!}\sum_{r=1}^{\infty} a_r^{-2k}\sum_{m=0}^{\infty}(-1)^m(m+k-1)\ldots(m+1)a_r^{-2m} \\
&= \frac{2}{(k-1)!}\sum_{m=0}^{\infty}(-1)^m(m+k-1)\ldots(m+1)\sum_{r=1}^{\infty} a_r^{-2(m+k)} \\
&= 2\sum_{m=0}^{\infty}(-1)^m\binom{m+k-1}{k-1}\sigma_{2(m+k)}\left(\frac{3}{2}\right),
\end{aligned}$$

where $\binom{n}{r}$ denotes the customary binomial coefficient and

$$\sigma_{2n}(\nu) = \sum_{r=1}^{\infty} j_{\nu r}^{-2n},$$

$j_{\nu r}$ being the r-th positive zero of $J_\nu(x)$. (The σ-notation is a modification due to Lehmer [7] of one used in [10] (p. 502). Still another notation is found in [3] (p. 61 (5).)

Thus,

$$\lambda_{2k}-1 = 2(-1)^k \sum_{r=k}^{\infty}(-1)^r \binom{r-1}{k-1} \sigma_{2r}\left(\frac{3}{2}\right) \tag{4}$$

$$= \frac{2(-1)^k}{(k-1)!} \sum_{r=k}^{\infty}(-1)^r (r-1)\ldots(r-k+1)\sigma_{2r}\left(\frac{3}{2}\right).$$

In particular,

$$\lambda_2 = 1-2\sum_{r=1}^{\infty}(-1)^r\sigma_{2r}\left(\frac{3}{2}\right).$$

Now ([3], p. 61 (4)) for $|x| < j_{\nu 1}$,

$$\frac{xJ_{\nu+1}(x)}{J_\nu(x)} = 2\sum_{r=1}^{\infty}\sigma_{2r}(\nu)x^{2r}. \tag{5}$$

It should be noted that $j_{\nu_1} > \nu$ ([10], p. 485 (1)). In particular, $j_{3/2,1} > 1$, a fact used implicitly in the derivation of formulas (7)-(10).

Hence, replacing x by $ix^{1/2}$, then dividing both sides by x, and denoting the resulting function by $f_\nu(x)$, we have, for $|x| < j^2_{\nu_1}$,

$$f_\nu(x) = \frac{I_{\nu+1}(x^{1/2})}{x^{1/2}I_\nu(x^{1/2})} = -2\sum_{r=1}^{\infty}(-1)^r\sigma_{2r}(\nu)x^{r-1}, \tag{6}$$

where $I_\nu(x)$ is the modified Bessel function of the first kind ([3], p. 5, (12) and [10], p. 77, (2)). Thus, (4) and (6) give Zygmund's result

$$\lambda_2 = \tfrac{1}{2}(e^2-5) = 1.194528049\ldots \tag{7}$$

on putting $x = 1$ in (6) and evaluating $f_{3/2}(1)$ ([10], p. 80 (10)).

Moreover, using (6) in combination with the second series in (4) provides an expression for λ_{2k} in closed form for all k, namely,

$$\lambda_{2k+2} = 1+\frac{(-1)^k}{k!}f^{(k)}_{3/2}(1), \quad k = 0, 1, \ldots, \tag{8}$$

in which (7) has been subsumed, following the usual convention of defining the zero-th derivative of a function to be the function itself, and where $f_{3/2}(x)$ is defined by (6).

3. A generating function. Representation (8) suggests a convenient way of constructing a generating function for the sequence $\lambda_2, \lambda_4, \ldots, \lambda_{2n}, \ldots$ From the Taylor expansion of $f_{3/2}(t)$ about the point 1 we have

$$f_{3/2}(t) = \sum_{k=0}^{\infty} \frac{f_{3/2}^{(k)}(1)}{k!}(t-1)^k = \sum_{k=0}^{\infty} \frac{(-1)^k f_{3/2}^{(k)}(1)}{k!}(1-t)^k$$

$$= \sum_{k=0}^{\infty} (\lambda_{2k+2}-1)(1-t)^k,$$

so that

$$f_{3/2}(1-x) = \sum_{k=0}^{\infty} (\lambda_{2k+2}-1)x^k,$$

where each series has a positive and finite radius of convergence.

Hence a generating function is

$$\frac{I_{5/2}(\sqrt{1-x})}{\sqrt{1-x}\, I_{3/2}(\sqrt{1-x})} = \sum_{k=0}^{\infty} (\lambda_{2k+2}-1)x^k \tag{9}$$

or

$$\frac{I_{5/2}(\sqrt{1-x})}{\sqrt{1-x}\, I_{3/2}(\sqrt{1-x})} + \frac{1}{1-x} = \sum_{k=0}^{\infty} \lambda_{2k+2}x^k. \tag{10}$$

4. Some numerical values. From (3) it is obvious that $\lambda_{2k} \downarrow 1$ as $k \uparrow \infty$. In (7) it is recorded that $\lambda_2 = 1.1945280\ldots$. From (6) and (8) it can be shown (using [10], p. 79, (3) and p. 80, (10)) that

$$\begin{cases} \lambda_4 = 1+\frac{1}{8}(e^4-4e^2-25) = 1.0052407\ldots, \\ \lambda_6 = 1+\frac{1}{32}(e^6-6e^4+3e^2-98) = 1.0002206\ldots, \\ \lambda_8 = 1+\frac{1}{384}(3e^8-24e^6+36e^4-8e^2-1167) = 1.000010077\ldots, \end{cases} \tag{11}$$

values already very close to 1.

Quite accurate upper bounds for λ_{2k} can be obtained readily by noticing from (3) that

$$1 < \lambda_{2k} < 1+2\sigma_{2k}\left(\frac{3}{2}\right), \tag{12}$$

and using the formulas for $\sigma_{2k}(\nu)$, given explicitly by Lehmer [7] for $k = 1, 2, \ldots, 12$, and in [10], p. 502, for $k = 1, 2, 3, 4, 5, 8$.

Inequality (12) then shows, e.g., that $\lambda_2 < 1.2$, quite close to the actual value given in (7) and more accurate than the best estimate $\lambda_2 \leqslant 1+2\pi^{-2}$ (due to Grace Chisholm Young) known before Zygmund's determination of λ_2. (For references to the earlier work, cf. [4], p. 224--225).

Additional bounds are

$$\lambda_4 < 1+\frac{1}{175} < 1.00572, \quad \lambda_6 < 1+\frac{2}{7875} < 1.000254,$$

$$\lambda_8 < 1+\frac{37}{3031875} < 1.0000123, \quad \lambda_{10} < 1+\frac{118}{36196875} < 1.0000033.$$

5. Comments and acknowledgements. (i) The general connection between Riemann summability and Bessel functions has been observed before. In fact, in 1943, Minakshisundaram [8] introduced what he called *Bessel summability*, replacing in fromula (1) the convergence factors $(na)^{-2k}\sin^{2k}na$ by

$$\{2^{\nu}\Gamma(\nu+1)\}^{K}\{\mu_n^{\nu}a^{\nu}\}^{-K}\{J_\nu(\mu_n a)\}^{K},$$

where $0 < \mu_1 < \mu_2 < \ldots \to \infty$.

He pointed out that this method becomes the Riemann method $(R, 2)$ on choosing $\mu_n = n, \nu = 1/2, K = 2$, the Lebesgue method on putting $\mu_n = n, \nu = 1/2, K = 1$.

From the point of view taken by Minakshisundaram the reason for the presence of the zeros $a_r = j_{3/2,r}$ in Zygmund's work becomes clearer. They yield the successive extrema of $x^{-\nu}J_\nu(x)$, $0 < x < \infty$, when $\nu = 1/2$, as shown by the familiar differentiation formula $D_x\{x^{-\nu}J_\nu(x)\} = -x^{-\nu}J_{\nu+1}(x)$.

This formula permits a ready formulation of the problem of determining for regular Bessel (Minakshisundaram) methods the constants analogous to λ_{2k}. In this case, they are

$$1+2^{\nu K+1}\{\Gamma(\nu+1)\}^{K}\sum_{r=1}^{\infty}|(j_{\nu+1,r})^{-\nu}J_\nu(j_{\nu+1,r})|^{K},$$

reducing to λ_{2k} when $\nu = \frac{1}{2}$, $K = 2k, k = 1, 2, \ldots$

(ii) The Rayleigh function $\sigma_{2n}(\nu)$ and related polynomials have been studied recently by Kishore ([5] and [6]).

(iii) This paper was written partly at Aarhus University, Denmark (where it benefited from useful remarks made by Z. Ciesielski during his visit there) and partly at the Royal Institute of Technology, Stockholm, Sweden (where M. Giertz and G. Welander were kind enough to help with the computations in § 4).

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Reçu par la Rédaction le 22. 4. 1965