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THE STRUCTURE OF CLOSURE CONGRUENCES

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The definition of a congruence for a closure space was introduced in [2] as the natural analogue of the usual concept of congruences for abstract algebras. The theory of congruences for abstract algebras has been focused primarily upon the underlying lattice structure (cf. the Grätzer-Schmidt representation theorem for algebraic lattices [3]), and in this note it will be shown that the compactness property for closures leads to a lattice structure on the congruences.

A *closure space* (C, S) is a set S with a closure operator C (i.e., C is a monotone, extensive and idempotent set mapping on S), and a *closure congruence* E for (C, S) is an equivalence relation on S which satisfies the set inclusion $CE(A) \subset EC(A)$ for all A contained in S. In this inequality E is interpreted to be the closure operator defined by $E(B) = \{y: \langle x, y\rangle \in E \text{ for some } x \in B\}$. The family of all closure congruences for (C, S) is denoted by $\mathfrak{K}(C, S)$. The most interesting structure on $\mathfrak{K}(C, S)$ has been obtained when C is *compact*, i.e., C satisfies $C(A) = \bigcup\{C(B): B \subset A, B \text{ finite}\}$ for all A contained in S. Compact closures result when one considers the subalgebra-closure of a finitary abstract algebra.

Let $\Pi(S)$ denote the family of all equivalence relations on S; with set inclusion as a partial ordering this becomes a complete lattice, so let $\vee$ and $\wedge$ denote the lattice operations. If R_i, $i \in I$, is an indexed family of relations on S, let $\bigcup\limits_{i\in I} R_i$ be the set-theoretic union of the R_i, and let $(\bigcup\limits_{i\in I} R_i)^n$ be the n-fold composition of this relation with itself. Then for $E_i \in \Pi(S)$, $i \in I$, there is the well-known expansion:

$$\bigvee_{i\in I} E_i = (\bigcup_{i\in I} E_i)^{\omega} \quad [= \bigcup_{n\in\omega} (\bigcup_{i\in I} E_i)^n].$$

Recalling that A is a *closed* subset of (C, S) iff $C(A) = A$, a useful condition for membership in $\mathfrak{K}(C, S)$ is established in the following

LEMMA. *Suppose $E \in \Pi(S)$. Then $E \in \mathfrak{K}(C, S)$ iff E (as a closure) maps closed subsets of (C, S) into closed subsets.*

Proof. If $E \in \mathfrak{K}(C, S)$ and A is a closed subset, then $CE(A) \subset EC(A) = E(A)$ implies $E(A)$ is closed. Conversely, if E is an equivalence relation which maps closed subsets into closed subsets, then for $A \subset S$

$$CE(A) \subset CEC(A) = EC(A).$$

THEOREM 1 (1). *Let* (C, S) *be a compact closure space and let* $E_i \in \mathfrak{K}(C, S)$ *for* $i \in I$. *Then* $\bigvee_{i \in I} E_i \in \mathfrak{K}(C, S)$, *and hence* $\mathfrak{K}(C, S)$ *is a complete lattice.*

Proof. Let E_i be as in the above hypothesis and suppose A is a closed subset of (C, S). By the lemma it will suffice to prove that $(\bigvee_{i \in I} E_i)(A)$ is a closed set. Let $i_1, \ldots, i_m$ be a finite subset of I. Then

$$E_{i_1} \vee \ldots \vee E_{i_m} = \bigcup_{n \in \omega} (E_{i_1} \ldots E_{i_m})^n,$$

and hence one easily sees that

$$(E_{i_1} \vee \ldots \vee E_{i_m})(A) = \Big(\bigcup_{n \in \omega} (E_{i_1} \ldots E_{i_m})^n\Big)(A) = \bigcup_{n \in \omega} [(E_{i_1} \ldots E_{i_m})^n (A)].$$

From the lemma it is immediate that $(E_{i_1} \ldots E_{i_m})^n(A)$ is closed for any $n \in \omega$, and since $\{(E_{i_1} \ldots E_{i_m})^n(A) \colon n \in \omega\}$ is a nest of sets, it follows from a theorem of Birkhoff and Frink on compact closures [1] that $\bigcup_{n \in \omega} (E_{i_1} \ldots E_{i_m})^n(A)$ is a closed set. Thus $E_{i_1} \vee \ldots \vee E_{i_m} \in \mathfrak{K}(C, S)$. Let $E_j^*, j \in J$, denote the family of all finite joins of the E_i, $i \in I$. Then clearly

$$\bigvee_{i \in I} E_i = \bigvee_{j \in J} E_j^* = \bigcup_{j \in J} E_j^*,$$

so if A is a closed subset of S, then

$$(\bigvee_{i \in I} E_i)(A) = (\bigcup_{j \in J} E_j^*)(A) = \bigcup_{j \in J} E_j^*(A).$$

$E_j^* \in \mathfrak{K}(C, S)$ for $j \in J$, hence $\{E_j^*(A)\}_{j \in J}$ is a family of closed sets directed upward, and appealing again to Birkhoff-Frink's Theorem, we have $(\bigvee_{i \in I} E_i)(A)$ is closed.

Although a description of all lattices of closure congruences is not known, the following representation theorem shows that a large class is certainly to be included.

THEOREM 2. *Let* $(P, \leqslant)$ *be a partially ordered set with zero. Then there is a closure* C *on* P *such that* $\mathfrak{K}(C, P)$ *is isomorphic to the lattice of lower ideals of* $(P, \leqslant)$.

Proof. Define a closure C on P by requiring that (i) the closure of a singleton is just a singleton, and (ii) the closure of any other set is the

(1) The proof of Theorem 1 is a refinement of a result in the author's Doctoral Dissertation, University of Oklahoma, 1968.

lower ideal generated by that set. Let $E \in \mathfrak{K}(C, P)$. Then $E(\{x\})$ is a closed set for each $x \in P$, hence the equivalence classes of E consist of single points and a lower ideal (only one since there is a zero!). It is easily seen that every such equivalence relation is indeed a congruence, and the isomorphism between congruences and lower ideals is straightforward.

Concluding remarks. One might hazard a guess that $\mathfrak{K}(C, S)$ is a sublattice of $\Pi(S)$ (as is the case with congruences of abstract algebras). However, one can find counterexamples on four element sets. Let $S = \{1, 2, 3, 4\}$, and let the closed subsets be $\emptyset$, $\{1\}$, $\{1, 2\}$, $\{1, 3\}$, S. Let E_1 and E_2 be the two equivalence relations whose equivalence classes are $\{1, 3\}$, $\{2, 4\}$ and $\{1\}$, $\{2, 3, 4\}$, respectively. It is easily verified that $E_1, E_2 \in \mathfrak{K}(C, S)$ (by applying the Lemma preceding Theorem 1); but $E_1 \cap E_2(\{1, 2\}) = \{1, 2, 4\}$, which shows that $E_1 \cap E_2$ does not map closed sets to closed sets, and hence $E_1 \cap E_2 \notin \mathfrak{K}(C, S)$.

With this example it is apparent that closure congruences are not going to be a direct translation of the theory of congruences for algebras. A rather striking observation in working with closure congruences is the fact that the structure of $\mathfrak{K}(C, S)$ may change radically if the compactness property does not hold — in particular the lattice structure may not exist; and even with a compact closure the lattice is not necessarily algebraic.

REFERENCES

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[3] G. Grätzer and E. T. Schmidt, *Characterization of congruence lattices of abstract algebras*, Acta Scientiarum Mathematicarum (Szeged) 24 (1963), p. 34-59.

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