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MEASURABLE LINEAR OPERATORS ON BANACH SPACES

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In the present paper we study measurable linear operators on Banach spaces which extend the concept of measurable linear functionals. Measurable linear functionals have been studied by a lot of authors, amongst others, Cameron and Graves [3], Šilov and Fan Dyk Tin [8], Gihman and Skorohod [4], Kanter [6] and [7], Hoffman-Jørgensen [5]. For the definition and the properties of measurable linear functionals on a real separable complete locally convex linear metric space we refer to paper [9]. Contrary to this, measurable linear operators have seldom been treated. The first authors who considered such operators in the case of the Wiener measure on the space of continuous functions were Šilov and Fan Dyk Tin (see [8]). Moreover, the theory of measurable linear operators on a separable Hilbert space has been presented in [4].

Our aim in this paper is to define and to investigate the concepts of Lusin operators and the operator Riesz property for probability measures on Banach spaces, analogously to those introduced by Urbanik in the case of measurable linear functionals (cf. [9]).

Let X denote a real separable Banach space with the norm $\|\cdot\|$ and with the dual space X^* and let μ be a Borel probability measure on X. We say that A is a *measurable linear operator* or, more precisely, A is a *μ-measurable linear operator* if A is a μ-measurable mapping from X into X which is defined on a μ-measurable linear manifold D_A with $\mu(D_A) = 1$ and for any pair $\alpha, \beta \in R$ and $x, y \in D_A$ the equality

$$A(\alpha x + \beta y) = \alpha A(x) + \beta A(y)$$

holds.

It is evident that each continuous linear operator on X is μ-measurable for every Borel probability measure μ.

Urbanik in [9] considered Lusin measurable linear functionals as limits of sequences of continuous linear functionals with respect to the convergence μ-almost everywhere (μ-a.e.). Analogously, it is possible to define Lusin measurable linear operators. Since in this case one can consider either weak or strong convergence of elements in a Banach space, we may define Lusin operators in the weak or strong sense.

DEFINITION 1. Let A be a μ-measurable linear operator on a real separable Banach space X. We say that A is a *weak* (*strong*) *Lusin operator* if there exists a sequence $\{A_n\}$ of continuous linear operators on X such that the sequence $\{A_n x\}$ is convergent weakly (strongly) to Ax μ-a.e., i.e., for every $f \in X^*$

$$f(A_n x) \to f(Ax) \ \mu\text{-a.e.} \qquad (\|A_n x - Ax\| \to 0 \ \mu\text{-a.e.}).$$

Obviously, if A is a strong Lusin operator, then A is also a weak Lusin operator.

Gihman and Skorohod ([4], p. 618) formulated the theorem which states that for each non-degenerate measure μ on a separable Hilbert space X, i.e., each measure vanishing on every proper subspace, each measurable linear operator is a strong Lusin operator. Thus they obtained also the equivalence of both the definitions of a Lusin operator. Unfortunately, the theorem of Gihman and Skorohod is not true. A simple counterexample, in the case of measurable linear functionals, is due to Kanter ([6], p. 447). Nevertheless, one can show that both the definitions of the Lusin operator on a Hilbert space are equivalent. Our purpose in the present note is to prove this equivalence, and even in a more general case, namely for Banach spaces with the approximation property.

A Banach space X is said to be a *space with the approximation property* if for every compact subset K of X and for every $\varepsilon > 0$ there exists a finite-dimensional continuous operator T on X such that $\|Tx - x\| < \varepsilon$ for any $x \in K$. It is easy to see that each Banach space with the Schauder basis has the approximation property (see [1], p. 514).

Now we are ready to prove the main result of the present paper.

THEOREM 1. *Let X be a real separable Banach space with the approximation property and let μ be a Borel probability measure on X. Suppose that A is a μ-measurable linear operator on X. Then A is a strong Lusin operator if and only if A is a weak Lusin operator.*

Proof. It is enough to show that a weak Lusin operator is strong Lusin since the converse implication is obvious. Let therefore A be a weak Lusin operator. To show that A is strong Lusin we prove that for any $\varepsilon > 0$ and $\varrho > 0$ there exists a continuous linear operator $\bar{A}$ on X such that

$$\mu\{x\colon \|Ax - \bar{A}x\| > \varepsilon\} < \varrho. \tag{1}$$

Let ν denote a probability measure on X given by the formula $\nu(E) = \mu(A^{-1}(E))$ for every Borel subset E of X. Since each Borel probability measure on X is tight (see [2], Theorem 1.4), there exists a compact subset K of X such that

$$\nu(X\setminus K) < \varrho/2.$$

Put $K' = A^{-1}(K)$. Then

$$\mu(X\setminus K') < \varrho/2. \tag{2}$$

Now, since X is a Banach space with the approximation property, there exists a finite-dimensional continuous operator T on X such that $\|y-Ty\| < \varepsilon/2$ for any $y\in K$. Hence

$$\|Ax-T(Ax)\| < \varepsilon/2 \quad \text{for any } x\in K'. \tag{3}$$

It is well known that each finite-dimensional operator T on X can be represented in the form

$$Ty = \sum_{k=1}^{m} f_k(y)\, y_k,$$

where $f_1, \ldots, f_m \in X^*$ and $\{y_1, \ldots, y_m\}$ is a linear basis in $T(X)$ with $\|y_k\| = 1$ for $k = 1, \ldots, m$ (see [1], p. 492).

Taking into account this representation, we can write inequality (3) in the form

$$\Big\|Ax-\sum_{k=1}^{m} f_k(Ax)\, y_k\Big\| < \varepsilon/2 \quad \text{for any } x\in K'.$$

Hence and from (2) we infer that

$$\mu\{x\colon \Big\|Ax-\sum_{k=1}^{m} f_k(Ax)\, y_k\Big\| > \varepsilon/2\} \leqslant \mu(X\setminus K') < \varrho/2. \tag{4}$$

Since, by the assumption, A is a weak Lusin operator, there exists a sequence $\{A_n\}$ of continuous linear operators on X such that for every $f\in X^*$

$$f(A_n x) \to f(Ax) \ \mu\text{-a.e.}$$

Hence for any $k = 1, \ldots, m$ there exists $n_k > 0$ such that

$$\mu\{x\colon |f_k(A_{n_k} x)-f_k(Ax)| > \varepsilon/2m\} < \varrho/2m. \tag{5}$$

Now we define a continuous linear operator $\bar{A}$ on X setting

$$\bar{A}(x) = \sum_{k=1}^{m} f_k(A_{n_k} x)\, y_k.$$

Then, combining (4) with (5), we have

$$\mu\{x:\ \|Ax-\bar{A}x\|>\varepsilon\}=\mu\{x:\ \|Ax-\sum_{k=1}^{m}f_k(A_{n_k}x)y_k\|>\varepsilon\}$$
$$\leqslant\mu\{x:\ \|Ax-\sum_{k=1}^{m}f_k(Ax)y_k\|>\varepsilon/2\}$$
$$+\mu\{x:\ \|\sum_{k=1}^{m}f_k(Ax)y_k-\sum_{k=1}^{m}f_k(A_{n_k}x)y_k\|>\varepsilon/2\}$$
$$<\varrho/2+\sum_{k=1}^{m}\mu\{x:\ |f_k(Ax)-f_k(A_{n_k}x)|>\varepsilon/2m\}$$
$$<\varrho/2+m\varrho/2m=\varrho.$$

Inequality (1) is thus proved.

Taking into account (1) and choosing the sequences $\varepsilon\to 0$ and $\varrho\to 0$ we can construct a sequence of continuous linear operators on X which is convergent in the measure μ to A, and from this sequence we may choose a subsequence which is strongly convergent to A μ-a.e. This completes the proof of the theorem.

Both the definitions of a Lusin operator are therefore equivalent. Further on we shall say directly a Lusin operator without specifying strong or weak.

Urbanik in [9] considered also the concept of the Riesz property for a Borel probability measure μ on a real separable complete locally convex linear metric space X. Namely, he said that such a measure has the *Riesz property* if every μ-measurable linear functional is a Lusin functional. It is obvious that each probability measure on a finite-dimensional linear space X has the Riesz property. The Riesz property has been studied by several authors. The first non-trivial result is due to Cameron and Graves [3] who proved that the Wiener measure considered on the space of continuous functions has the Riesz property. This theorem was strengthened by Kanter ([6], p. 448) who proved that each Gaussian measure with zero mean has the Riesz property. Moreover, Urbanik [9] and Kanter [7] studied the Riesz property for probability measures on the L^2-space over the unit interval, induced by a symmetric, homogeneous, separable stochastic process with independent increments, which is continuous in probability.

Analogously to the case of measurable linear functionals we define the operator Riesz property.

DEFINITION 2. Let X be a real separable Banach space. We say that a Borel probability measure μ on X has the *operator Riesz property* if every μ-measurable linear operator is a Lusin operator.

However, it appears that in the case of Banach spaces with the approximation property the operator Riesz property yields nothing of importance.

Namely, a slight modification of the proof of Theorem 1 shows the following statement:

THEOREM 2. *Let μ be a Borel probability measure on a real separable Banach space X with the approximation property. Then μ has the operator Riesz property if and only if μ has the (functional) Riesz property.*

REFERENCES

[1] G. P. Akilov and L. V. Kantorovič, *Functional Analysis* (in Russian), Moscow 1977.
[2] P. Billingsley, *Convergence of Probability Measures*, New York 1968.
[3] R. H. Cameron and R. E. Graves, *Additive functionals on a space of continuous functions. I*, Trans. Amer. Math. Soc. 70 (1951), pp. 160–176.
[4] I. I. Gihman and A. V. Skorohod, *A Theory of Random Processes. I* (in Russian), Moscow 1971.
[5] J. Hoffman-Jørgensen, *Integrability of seminorms, the 0-1 law and the affine kernel for product measures*, Studia Math. 61 (1977), pp. 137–159.
[6] M. Kanter, *Linear sample spaces and stable processes*, J. Func. Anal. 9 (1972), pp. 444–459.
[7] — *Completion measurable linear functionals on a probability space*, Colloq. Math. 38 (1978), pp. 277–304.
[8] G. E. Šilov and Fan Dyk Tin, *Integral, Measure and Derivative in Linear Spaces* (in Russian), Moscow 1967.
[9] K. Urbanik, *Random linear functionals and random integrals*, Colloq. Math. 33 (1975), pp. 255–263.

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