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ON FOURIER SERIES WITH RESPECT TO THE BESSEL POLYNOMIAL-SYSTEM

BY

R. TABERSKI (POZNAŃ)

1. Introduction. It is known [1] that the differential equation

$$z^2\frac{d^2\omega}{dz^2}+2(z+1)\frac{d\omega}{dz}=n(n+1)\omega,$$

where n is an arbitrary non-negative integer, has as a special solution the Bessel polynomial:

$$\omega_n(z)=\sum_{r=0}^{n}\frac{(n+r)!}{r!(n-r)!}\left(\frac{z}{2}\right)^r=\frac{1}{2^n}e^{2/z}\frac{d^n}{dz^n}(z^{2n}e^{-2/z}).$$

These polynomials form an orthogonal system with the weight function $e^{-2/z}$ on the unit circle $U=\{z\colon |z|=1\}$, that is

$$\int_U \omega_p(z)\omega_q(z)e^{-2/z}dz=0 \quad \text{if} \quad p\neq q.$$

Moreover,

$$\frac{1}{2\pi i}\int_U \omega_n^2(z)e^{-2/z}dz=(-1)^{n+1}\frac{2}{2n+1} \quad \text{for} \quad n=0,1,2,\ldots$$

Here and below the integration is counter-clockwise along U.

Let $f(z)$ be a complex-valued function continuous on U; write

$$c_n=\int_U f(z)\omega_n(z)e^{-2/z}dz\Big/\int_U \omega_n^2(z)e^{-2/z}dz.$$

Then the series

$$\sum_{n=0}^{\infty}c_n\omega_n(z) \tag{1}$$

is called the *Fourier series* of f.

In the present note we give two theorems concerning the uniform and absolute convergence of Fourier series in the unit disc.

2. Fundamental lemmas. Let us begin with the following lemma:

2.1. *The inequalities*

$$\frac{(2n)!}{2^n n!} \leqslant \max_{|z|\leqslant 1} |\omega_n(z)| < e\,\frac{(2n)!}{2^n n!}$$

hold for $n = 0, 1, 2, \ldots$

Proof. By the maximum modulus principle,

$$\max_{|z|\leqslant 1} |\omega_n(z)| = \max_{|\varphi|\leqslant \pi} |\omega_n(e^{i\varphi})|.$$

The polynomial $\omega_n(z)$ has positive coefficients. Hence the last maximum is equal to $\omega_n(1)$. Since

$$\omega_n(1) = \sum_{r=0}^{n} \frac{(n+r)!}{r!(n-r)!}\cdot\frac{1}{2^r} = e\left(\frac{2}{\pi}\right)^{1/2} K_{n+1/2}(1),$$

where

$$K_\nu(x) = \frac{\sqrt{\pi}}{\Gamma(\nu+1/2)}\left(\frac{x}{2}\right)^{\nu}\int_1^\infty e^{-xt}(t^2-1)^{\nu-1/2}dt$$

(see [3], § 3.71, [2], p. 153), we have

$$\omega_n(1) = \frac{e}{2^n n!}\int_1^\infty e^{-t}(t^2-1)^n dt.$$

Observing that

$$\frac{1}{e}\int_0^\infty e^{-s}s^{2n}\,ds \leqslant \int_1^\infty e^{-t}(t^2-1)^n dt < \int_0^\infty e^{-t}t^{2n}dt,$$

$$\int_0^\infty e^{-x}x^{2n}dx = (2n)!,$$

we get the estimate as desired.

Let $S_m(z;f)$ be the m-th partial sum of Fourier series (1) of f, i. e.

$$(2)\qquad S_m(z;f) = \sum_{n=0}^{m} c_n\omega_n(z) = \frac{1}{2\pi i}\int_U f(t)\Phi_m(t,z)e^{-2/t}dt,$$

where

$$(3) \qquad \Phi_m(t, z) = \frac{1}{2}\sum_{n=0}^{m}(-1)^{n+1}(2n+1)\,\omega_n(t)\,\omega_n(z).$$

By the formula ([1], p. 104-105)

$$\frac{1}{2\pi i}\int_U \omega_n(t)e^{-2/t}dt = \begin{cases} -2 & \text{if} \quad n = 0, \\ 0 & \text{if} \quad n = 1, 2, \ldots, \end{cases}$$

the Fourier coefficients of $f(z) \equiv 1$ are $c_0 = 1$ and $c_n = 0$ for $n \geqslant 1$. Thus

2.2. *We have*

$$\frac{1}{2\pi i}\int_U \Phi_m(t, z)e^{-2/t}dt = 1$$

for any complex z *and* $m = 0, 1, 2, \ldots$

Finally, a lemma of Christoffel's type will be given.

2.3. *The kernel* (3) *can be written in the form*

$$\Phi_m(t, z) = (-1)^{m+1}\frac{\omega_{m+1}(t)\,\omega_m(z) - \omega_{m+1}(z)\,\omega_m(t)}{2(t-z)}$$

for each complex t, z, $t \neq z$.

Proof. Applying the reccurence relation ([1], p. 101)

$$\omega_{n+1}(x) - (2n+1)x\omega_n(x) - \omega_{n-1}(x) = 0 \qquad (n \geqslant 1),$$

we get

$$(t-z)(-1)^{n+1}(2n+1)\,\omega_n(t)\,\omega_n(z) = (-1)^{n+1}\{\omega_{n+1}(t)\,\omega_n(z) - \\ - \omega_{n+1}(z)\,\omega_n(t)\} - (-1)^n\{\omega_n(t)\,\omega_{n-1}(z) - \omega_n(z)\,\omega_{n-1}(t)\},$$

whence

$$(t-z)\sum_{n=1}^{m}(-1)^{n+1}(2n+1)\,\omega_n(t)\,\omega_n(z)$$

$$= (-1)^{m+1}\{\omega_{m+1}(t)\,\omega_m(z) - \omega_{m+1}(z)\,\omega_m(t)\} + \{\omega_1(t)\,\omega_0(z) - \omega_1(z)\,\omega_0(t)\}.$$

Observing that the expression in the last braces equals to $t - z$, we obtain the required formula for $\Phi_m(t, z)$.

3. Main results. Now we shall present the following theorem:

3.1. *If* $\sum_{n=0}^{\infty} |a_n| < \infty$, *then the Fourier series* (1) *of the function*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{4}$$

converges to $f(z)$ *uniformly in the disc* $|z| \leqslant 1$.

Proof. Suppose that $|z| < 1$. Using notation (2), (3) and lemmas 2.2 and 2.3, we have

$$\begin{aligned} S_m(z;f)-f(z) &= \frac{1}{2\pi i}\int_U \{f(t)-f(z)\}\Phi_m(t,z)e^{-2/t}dt \\ &= \frac{(-1)^{m+1}}{4\pi i}\Big\{\omega_m(z)\int_U g(t,z)\,\omega_{m+1}(t)e^{-2/t}dt- \\ &\qquad -\omega_{m+1}(z)\int_U g(t,z)\,\omega_m(t)e^{-2/t}dt\Big\} \end{aligned}$$

with

$$g(t,z) = \{f(t)-f(z)\}/(t-z).$$

The identity

$$g(t,z) = \sum_{n=1}^{\infty} a_n \frac{t^n-z^n}{t-z} = \sum_{n=1}^{\infty} a_n (t^{n-1}+t^{n-2}z+\ldots+z^{n-1})$$

implies

$$I_m(z) \equiv \int_U g(t,z)\,\omega_m(t)e^{-2/t}dt = \sum_{n=1}^{\infty} a_n \Big(\sum_{k=0}^{n-1} z^{n-1-k}\int_U t^k\omega_m(t)e^{-2/t}dt\Big).$$

Since ([1], p. 104-105)

$$\frac{1}{2\pi i}\int_U t^k\omega_m(t)e^{-2/t}dt = \begin{cases} 0 & \text{if} \quad k < m, \\ \dfrac{(-2)^{k+1}k!}{(k-m)!(m+k+1)!} & \text{if} \quad k \geqslant m \end{cases}$$

and

$$\sum_{p=1}^{q} \frac{2^{m+p}(m+p-1)!}{(p-1)!(2m+p)!} < \frac{2^{m+1}m!}{(2m+1)!}e^2 \quad \text{for any } q \geqslant 1,$$

we get

$$\left|\frac{1}{4\pi i}I_m(z)\right| \leqslant e^2\frac{2^m m!}{(2m+1)!}\lambda_m,$$

where $\lambda_m = \sum_{n=m+1}^{\infty} |a_n|$. This inequality together with 2.1 gives

$$|S_m(z;f)-f(z)| \leqslant \frac{1}{4\pi}\{|\omega_m(z)||I_{m+1}(z)|+|\omega_{m+1}(z)||I_m(z)|\}$$

$$\leqslant e^3\left\{\frac{\lambda_{m+1}}{(2m+1)(2m+3)}+\lambda_m\right\} \quad \text{for} \quad |z|<1.$$

By Abel's theorem, the last estimate of $|S_m - f|$ remains true if $|z| = 1$, and thus the proof is completed.

Observing that

$$(5) \qquad |c_n\omega_n(z)| \leqslant |S_n(z;f)-f(z)|+|f(z)-S_{n-1}(z;f)| \leqslant 3e^3\lambda_{n-1} \quad \text{when} \quad |z|\leqslant 1,$$

and using Abel's transformation, we obtain

3.2. *If* $\sum_{n=1}^{\infty} n|a_n| < \infty$, *then the Fourier series* (1) *of* (4) *converges absolutely for* $|z| \leqslant 1$.

Notice that if $f(z)$ is regular and bounded in the disc $|z| < R$, $R > 1$, then there exists a constant A such that

$$|S_m(z;f)-f(z)| \leqslant A/R^m \quad \text{for} \quad |z|\leqslant 1,\ m = 0,1,2,\ldots$$

This result follows at once from (5) and the Cauchy inequality $|a_n| \leqslant M/R^n$ ($M = \text{const}$).

4. Appendix. We shall point out the generating functions for the polynomials $\omega_n(z)$ and their generalizations.

The power u^v ($u \neq 0$) will denote $\exp(v\operatorname{Log}u)$, where $\operatorname{Log}u$ is the principal value of $\log u$.

4.1. *We have*

$$(6) \qquad (1-2zt)^{-1/2}\exp\left\{\frac{1-(1-2zt)^{1/2}}{z}\right\} = \sum_{n=0}^{\infty}\frac{\omega_n(z)}{n!}t^n$$

for each complex z, t *such that* $z \neq 0$, $|zt| < 1/2$.

Proof. The inequality $|zt| < 1/2$ implies $\operatorname{Re}(1-2zt) > 0$. Denoting the left-hand side of (6) by $Q(t, z)$, we can write

$$Q(t,z) = \sum_{n=0}^{\infty}\varphi_n(z)t^n$$

in the disc $D_z = \{t\colon |t| < 1/|2z|\}$, $z \neq 0$.

Let C be a contour surrounding the point $t = 0$, $C \subset D$. Then

$$\varphi_n(z) = \frac{1}{2\pi i}\int_C \frac{Q(t,z)}{t^{n+1}}dt.$$

Substituting

$$\frac{1-(1-2zt)^{1/2}}{z} = \frac{2}{z} - \frac{2}{u},$$

we obtain

$$\varphi_n(z) = \frac{1}{2\pi i}\int_{C'} \frac{u^{2n}}{2^n(u-z)^{n+1}}\exp\left(\frac{2}{z}-\frac{2}{u}\right)du,$$

where C' is a suitable contour surrounding $u = z$.

The residue of the last integrand at $u = z$ is equal to

$$\frac{1}{2^n n!}e^{2/z}\left[\frac{d^n}{du^n}(u^{2n}e^{-2/u})\right]_{u=z}.$$

Consequently,

$$\varphi_n(z) = \omega_n(z)/n!,$$

and expansion (6) is established (cf. [1], p. 106).

Analogously, considering the generalized Bessel polynomials

$$\omega_n^*(z) = \sum_{r=0}^{n}\binom{n}{r}(n+r+a-2)^{(r)}\left(\frac{z}{b}\right)^r,$$

where $a, b \neq 0$ are real constants and $(l)^{(r)}$ means $l(l-1)\ldots(l-r+1)$, we easily get

4.2. *The formula*

$$\frac{2^{a-2}}{(1-4zt/b)^{1/2}}\{1+(1-4zt/b)^{1/2}\}^{2-a}\exp\left\{b\,\frac{1-(1-4zt/b)^{1/2}}{2z}\right\} = \sum_{n=0}^{\infty}\frac{\omega_n^*(z)}{n!}t^n$$

holds, provided that $z \neq 0$, $|4zt| < |b|$ (see [1], p. 107-114).

REFERENCES

[1] H. L. Krall and O. Frink, *A new class of orthogonal polynomials: the Bessel polynomials*, Transactions of the American Mathematical Society 65 (1949), p. 100-115.

[2] Н. Н. Лебедев, *Специальные функции и их приложения*, Москва-Ленинград 1963.

[3] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge 1922.

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