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A CHARACTERIZATION OF LOCALLY COMPACT FIELDS, III

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It is well known that the only non-discrete locally compact topological fields are the real number field, p-adic number fields, the fields of formal Laurent series of one variable over finite fields and their finite extensions. In this note we shall give a short proof of the above-mentioned theorem.

THEOREM 1. *Let $(K, \mathcal{T})$ be a non-discrete locally bounded complete topological field with a non-zero topological nilpotent. Then*

(1) *if the characteristic of K is zero, then either Q lies discretely in K, or K contains as a closed topological subfield the reals $\boldsymbol{R}$ or a p-adic number field Q_p;*

(2) *if the characteristic of K is non-zero, then K contains as a closed topological subfield a field of formal Laurent series over a finite field.*

Remark. The first part of Theorem 1 remains true without the assumption that K has a non-zero topological nilpotent ([7], Theorem 3, p. 887). However, the original proof is more complicated than the proof presented here.

LEMMA 1 ([1], Theorem 6.1, p. 165). *Let $(K, \mathcal{T})$ be a non-discrete topological field. Then the topology is induced by a pseudonorm if and only if $(K, \mathcal{T})$ is locally bounded and in K there exists a non-zero topological nilpotent.*

LEMMA 2 ([4], (2.11) Satz, p. 256, and (2.13) Satz, p. 257). *Let K be the rational number field Q or the rational function field $\boldsymbol{F}_q(t)$ over a Galois field $\boldsymbol{F}_q$. Then every non-trivial pseudonorm on K is equivalent to a pseudonorm of the form* $\sup\{\varphi_i: i = 1, 2, \ldots, n\}$, *where φ_i are non-trivial norms on K.*

Proof of Theorem 1. Lemma 1 implies that $\mathcal{T}$ is induced on K by a non-trivial pseudonorm, say p.

Let K has zero characteristic and suppose that Q is a non-discrete subfield of K. It means that $p_1 = p\,|\,Q$ is a non-discrete pseudonorm inducing the original topology $\mathcal{T}$ on Q. But then, applying Lemma 2, we see that p_1 is equivalent to a pseudonorm of the form $\sup\{\varphi_i: i=1,2,\ldots,n\}$, where φ_i are non-discrete norms on Q. Since K is complete in $\mathcal{T}$, it must contain the closure of Q with respect to p_1, i.e. the completion of Q in p_1. Thus, by the Artin-Whaples approximation theorem, K contains as a topological subring the direct sum $\hat{Q}_1 \oplus \hat{Q}_2 \oplus \ldots \oplus \hat{Q}_n$, where $\hat{Q}_i$ is the completion of Q in the norm φ_i. However, according to Ostrowski's theorem, the ordinary absolute value and the p-adic norm are the only non-discrete norms on Q. Since a field has no proper zero divisors, the direct sum reduces to a field Q_p of the p-adic numbers or to the reals $\boldsymbol{R}$.

Now let the characteristic of K be non-zero. As in the previous case, the topology $\mathcal{T}$ is induced on K by a non-trivial pseudonorm p. Let $t \in K$ be a non-zero topological nilpotent. The element t is transcendental over the prime field $\boldsymbol{F}_p$ of K since, otherwise, it would be algebraic over $\boldsymbol{F}_p$ and, consequently, a root of unity, which is impossible, t being a nilpotent. It proves that K contains a subfield $F = \boldsymbol{F}_p(t)$ of the rational functions of one variable t over $\boldsymbol{F}_p$. Since t is a non-zero nilpotent element, the pseudonorm $p_1 = p \mid F$ is non-trivial. Lemma 2 implies that p_1 is equivalent to a pseudonorm of the form $\sup\{\varphi_i: i = 1, 2, \ldots, n\}$, where φ_i are norms. But every non-discrete norm on F is equivalent either to a q-adic norm ($q \in \boldsymbol{F}_p[t]$ — a prime element) or to the norm φ_∞, where

$$\varphi_\infty\left(\frac{f}{g}\right) = \exp[\deg(g) - \deg(f)] \quad (f, g \in \boldsymbol{F}_p[t]).$$

It implies that K contains the closure of F, i.e. the completion $\hat{F}$ of F in $\sup\{\varphi_i: i = 1, 2, \ldots, n\}$. The Artin-Whaples approximation theorem now gives

$$\hat{F} = \hat{F}_1 \oplus \hat{F}_2 \oplus \ldots \oplus \hat{F}_n,$$

where $\hat{F}_i$ is the completion of F at φ_i. Since every $\hat{F}_i$ is a formal Laurent series field over $\boldsymbol{F}_p$ and since K is a field, $n = 1$ and the proof is completed.

LEMMA 3. *Let $(K, \mathcal{T})$ be a non-discrete locally compact topological field. Then*

(1) *there exists a non-zero topological nilpotent in K;*

(2) *$\mathcal{T}$ is the topology of type V (i.e. for every neighbourhood U of zero the set $(K \setminus U)^{-1}$ is bounded);*

(3) *if K has zero characteristic, then $\mathcal{T} \mid Q$ is non-discrete.*

For completeness of the main result we shall present detailed proofs.

Proof. Let $\mathfrak{A}$ be a base of the neighbourhoods of zero in $(K, \mathcal{T})$.

(1) Let us take relatively compact $U \in \mathfrak{A}$ and let $V \in \mathfrak{A}$ be a proper subset of U. Since $\bar{U}$ is compact, $\bar{U}$ is bounded, so there exists $W \in \mathfrak{A}$ satisfying $UW \subset \bar{U}W \subset V$. Since $(K, \mathcal{T})$ is non-discrete, we can find

a non-zero $x \in W$ with

$$(*) \qquad \bar{U} \supset U \underset{\neq}{\supset} \bar{U}x \underset{\neq}{\supset} \ldots \underset{\neq}{\supset} \bar{U}x^n \supset Ux^n \underset{\neq}{\supset} \bar{U}x^{n+1} \supset \ldots$$

Let y be a cluster point of the set $\{x^n: n \in \boldsymbol{N}\}$. Then

$$\bigcap_{n \in \boldsymbol{N}} \bar{U}x^n = \bar{U}y.$$

We are going to show that $y = 0$. Otherwise, since $\bar{U}y$ is compact, we should have

$$\bigcap_{n \in \boldsymbol{N}} \bar{U}x^n = \bigcap_{n \in \boldsymbol{N}} (\bar{U}x)x^n = (\bar{U}x)y.$$

Thus $\bar{U}xy = \bar{U}y$ and $y \neq 0$ would imply $\bar{U}x = \bar{U}$, which contradicts $(*)$.

(2) Since $(K, \mathscr{T})$ is a locally bounded topological field, we have $\mathfrak{A} = \{aU: a \in K^*\}$ for every neighbourhood U of zero. We can suppose that U is an almost-order, i.e. $UU \subset U$, $0, 1 \in U$, $K^* = U^*(U^*)^{-1}$, and $z(U+U) \subset U$ for suitable $z \in U$, $z \neq 0$. Hence it is sufficient to show that $(K \setminus U)^{-1}$ is bounded. Let $x \in K$ be a non-zero topological nilpotent. Suppose, on the contrary, that $(K \setminus U)^{-1}$ is not bounded. Then there exists a sequence (a_n) such that both $a_n^{-1}x^n$ and a_n lie in $K \setminus U$. Putting $b_n = a_n^{-1}x^n$ we have $a_n b_n = x^n$. Let us note that for every $a \in K^*$ there is a smallest number $t(a)$ with $ax^{t(a)} \in U$. Then

$$ax^{t(a)-1} \in K \setminus U \quad \text{and} \quad ax^{t(a)} \in K \setminus xU.$$

Finally,

$$ax^{t(a)} \in U \cap (K \setminus xU).$$

Now let us take a_n (respectively, b_n) for a and put

$$c_n = a_n x^{t(a_n)}, \quad d_n = b_n x^{t(b_n)}, \quad A = U \cap (K \setminus xU).$$

Since U is bounded, $\bar{A}$ is compact and the sequences (c_n) and (d_n) have some cluster points in $\bar{A}$, say c and d. Since $c_n, d_n \in K \setminus xU$, the cluster points c and d are both non-zero. But the sequence

$$c_n d_n = (a_n x^{t(a_n)})(b_n x^{t(b_n)}) = x^{n+t(a_n)+t(b_n)}$$

tends to zero, which contradicts the continuity of the multiplication in $(K, \mathscr{T})$ ([5], Satz 8, p. 265).

(3) Suppose that $\mathscr{T} | Q$ is discrete. We shall prove that this case never appears. Let x be a non-zero topological nilpotent in K, and U — a relatively compact (hence bounded) neighbourhood of zero in K. We can suppose that U is an almost-order defining the topology $\mathscr{T}$ in K. Then the sets $x^n U$ ($n \in \boldsymbol{N}$) form a base of the neighbourhoods of zero in K. If $\mathscr{T} | Q$ were discrete, then Q^* would be disjoint with some $x^n U$, i.e.

$Q^* \subset K \setminus x^n U$. It means that

$$Q^* = (Q^*)^{-1} \subset (K \setminus x^n U)^{-1}.$$

Since the topology $\mathcal{T}$ is of type V, $(K \setminus x^n U)^{-1}$ is bounded, and so is Q^*. Thus, there exists $m \in \boldsymbol{N}$ with $Q^* x^m \subset Ux$, $Q^* \subset Ux^{1-m}$, or $Q^* \subset \overline{U} x^{1-m}$. Since $\overline{U}$ is compact, the inclusion $Q^* \subset \overline{U} x^{1-m}$ would imply that the set $\boldsymbol{N} \subset Q^*$ has a convergent subsequence. This contradicts the assumption that $\mathcal{T} | Q$ is discrete ([3], Hilfssatz 5, p. 66).

LEMMA 4. *Every locally compact topological vector space over a normed complete field is finite dimensional.*

For an elementary proof see [8], Theorem 3.6, p. 23.

THEOREM 2. *Let K be a field and let $(K, \mathcal{T})$ be a non-discrete topological ring. Then the following conditions are equivalent*:

(1) *K is a locally compact topological ring*;

(2) *K is a locally compact topological field*;

(3) *K is a finite extension of a field of one of the following types*: $\boldsymbol{R}$, Q_p, $\boldsymbol{F}_p\langle X\rangle$, $\boldsymbol{F}_p\{X\}$ (see [10] for the last two).

Proof. (3) $\Rightarrow$ (1). Since $\boldsymbol{R}$, Q_p, $\boldsymbol{F}_p\langle X\rangle$, and $\boldsymbol{F}_p\{X\}$ are all locally compact in the topologies induced by suitable norms, and they are complete, the only topologies on their finite extensions are the product topologies. This implies that the finite extensions of the above-mentioned fields are locally compact rings in the product topology.

(1) $\Leftrightarrow$ (2). Indeed, the implication (2) $\Rightarrow$ (1) is trivial and (1) $\Rightarrow$ (2) follows from [2], since the multiplicative group K^* of K is a locally compact space and since multiplication is continuous (so division must also be continuous).

(2) $\Rightarrow$ (3). Since K is locally compact and non-discrete, it has a neighbourhood U of zero such that $\overline{U}$ is compact. But any compact set is bounded, so the topology $\mathcal{T}$ is locally bounded. Moreover, in K there exists a non-zero topological nilpotent (Lemma 3). Lemma 1 implies that the topology $\mathcal{T}$ is induced in K by a non-trivial pseudonorm. Since $(K, \mathcal{T})$ is locally compact, in the case of zero characteristic the topology $\mathcal{T}$ is non-discrete on Q. Thus Theorem 1 implies that the reals, a p-adic number field, or the Laurent series field over a finite field is a closed topological subfield of K. Finally, it follows from Lemma 4 that K is a finite extension of one of the above-mentioned fields.

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