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# REMARKS ON C-INDEPENDENCE IN CARTESIAN PRODUCTS OF ABSTRACT ALGEBRAS

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A subset $B$ of an abstract algebra $\mathfrak{A}$ is *C-independent* if no $b \in B$ belongs to the subalgebra $C(B \setminus \{b\})$ of $\mathfrak{A}$ generated by $B \setminus \{b\}$.

The purpose of this paper is to explain relations between some kinds of independence on axes and the $C$-independence of subsets of products of abstract algebras. We also give examples which show that none of our results can be strengthened.

In section 3 we show that products of algebras do not preserve condition of exchange of independent elements.

For terminology and notations see [2].

**1. $C$-independence.** First let us state an obvious lemma without proof.

LEMMA 1.1. *Let $h$ be a homomorphism of an algebra $\mathfrak{A}$ onto $\mathfrak{B}$. If $\{b_0, \ldots, b_{n-1}\}$ is C-independent in $\mathfrak{B}$, and $h(a_i) = b_i$ for $i = 0, \ldots, n-1$, then $\{a_0, \ldots, a_{n-1}\}$ is C-independent in $\mathfrak{A}$.*

Now let $\langle \mathfrak{A}_t \rangle_{t \in T}$ $(T \neq \emptyset)$ be a system of similar algebras and let $\mathfrak{A}$ be their Cartesian product. Let us fix $t_0 \in T$ and let $a_0, \ldots, a_{n-1} \in A_{t_0}$. Put

$$A^{(i)} = \{a_i\} \times \prod_{t \in T \setminus \{t_0\}} A_t \quad \text{for } i = 0, \ldots, n-1.$$

Let $\{p_0, \ldots, p_{n-1}\}$ be a subset of $\prod_{t \in T} A_t$ such that $p_i \in A^{(i)}$ for $i = 0, \ldots, n-1$. Every set constructed in this way is called a *selector* of $\{a_0, \ldots, a_{n-1}\}$ in the product $\prod_{t \in T} A_t$.

PROPOSITION 1.2. *Let $\mathfrak{A} = \prod_{t \in T} \mathfrak{A}_t$ $(T \neq \emptyset)$. For an arbitrary index $t_0 \in T$ choose a set $\{a_0, \ldots, a_{n-1}\}$ C-independent in $\mathfrak{A}_{t_0}$ and let $\{p_0, \ldots, p_{n-1}\}$ be a selector of $\{a_0, \ldots, a_{n-1}\}$ in $A$. Then $\{p_0, \ldots, p_{n-1}\}$ is C-independent in $\mathfrak{A}$.*

Proof follows from Lemma 1.1, because projections are homomorphisms.

PROPOSITION 1.3. *Let $\mathfrak{A}$ be an algebra and let $\{a_0, \ldots, a_{n-1}\}$ be independent in $\mathfrak{A}$ and $\{b_0, b_1\}$ be C-independent in $\mathfrak{A}$. Then $\{(a_0, b_0), \ldots, (a_{n-1}, b_0), (a_{n-1}, b_1)\}$ is C-independent in $\mathfrak{A}^2$.*

Proof. Suppose that

(a) $a_i = f(a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n-1}, a_{n-1})$ for $i \neq n-1$ and $b_0 = f(b_0, \ldots, b_0, b_1)$.

The first equality contradicts our assumption that $\{a_0, \ldots, a_{n-1}\}$ is independent in $\mathfrak{A}$.

(b) $a_{n-1} = f(a_0, \ldots, a_{n-1})$, $b_0 = f(b_0, \ldots, b_0, b_1)$.

Using independence of $\{a_0, \ldots, a_{n-1}\}$, we have $f = e_n^{(n)}$, but this is impossible in view of $b_0 \neq b_1$.

(c) $a_{n-1} = f(a_0, \ldots, a_{n-1})$, $b_1 = f(b_0, \ldots, b_0)$.

Since $\{b_0, b_1\}$ is $C$-independent in $\mathfrak{A}$, this is impossible.

It is also easy to verify the following proposition:

PROPOSITION 1.4. *Let $\mathfrak{A} = \prod_{t \in T} A_t$ and let $P$ be a subset of $A$ such that, for each $t \in T$, its projection $\pi_t(P)$ is C-independent in $\mathfrak{A}_t$. Then each subset $Q \subset P$ satisfying the condition*

$$\mathop{\forall}_{p \in Q} \mathop{\exists}_{t \in T} \mathop{\forall}_{q \in Q} [p \neq q \Rightarrow p(t) \neq q(t)]$$

*is C-independent in $\mathfrak{A}$.*

In some cases construction of $C$-independent sets described in Proposition 1.4 gives maximal $C$-independent sets. This is shown in the following example:

EXAMPLE 1.5. Let $\mathfrak{A} = (0, 1, 2, 3; f)$, where the operation $f$ is defined as follows:

$f(x, y, z) =$ the fourth element of $A$ if $x, y, z$ are all different,

$$f(x, x, y) = f(x, y, x) = f(y, x, x) = x.$$

In the algebra $\mathfrak{A}$ each three-element subset is $C$-independent. Let us consider $\mathfrak{A}^2$ and let $Q = \{(0,0), (1,0), (2,1), (2,2)\}$. It is clear that $Q$ satisfies the hypothesis of Proposition 1.4 (for $P$ we set $\{0, 1, 2\} \times \{0, 1, 2\}$) and that $Q$ is a maximal $C$-independent subset of $\mathfrak{A}^2$.

More $C$-independent sets can be obtained in the Cartesian square of a given algebra if we put a stronger assumption on the set $P$.

THEOREM 1.6. *Let $P_1$ and $P_2$ be independent sets in an algebra $\mathfrak{A}$. Then each subset $Q$ of $P_1 \times P_2$ satisfying the condition: if $(x_i, y_j)$ and $(x_i, y_k)$ belong to $Q$, then either $(x, y_j) \in Q \Rightarrow x = x_i$ or $(x, y_k) \in Q \Rightarrow x = x_i$, is C-independent in $\mathfrak{A}^2$.*

Proof. Suppose that $(x, y) \in Q$ and $(x, y) \in C(Q \setminus \{(x, y)\})$. Then there is an algebraic function $f$ such that

$$(x, y) = f((x_1, y_1), \ldots, (x_n, y_n)),$$

where $(x_i, y_i) \in Q \setminus \{(x, y)\}$. In other words, $x = f(x_1, \ldots, x_n)$ and $y = f(y_1, \ldots, y_n)$. Of course, it must be $x \in \{x_1, \ldots, x_n\}$, since the set $\{x, x_1, \ldots, x_n\}$ is independent in $\mathfrak{A}$ as a subset of independent set $P_1$. Thus we have

$$(*) \quad x = f(x_1, \ldots, x_{q_1}, \underbrace{x, \ldots, x}_{r_1 \text{ times}}, x_{q_1+r_1+1}, \ldots, x_{q_m}, \underbrace{x, \ldots, x}_{r_m \text{ times}}, x_{q_m+r_m+1}, \ldots),$$

where all indexed arguments are different from $x$. From the definition of $Q$ there is an index $s$ of the form $s = q_i + j_i$ (where $j_i \leqslant r_i$) such that $y_s \neq y_t$ if $(x_t, y_t) \in Q$. This means that in the equation

$$(**) \qquad y = f(y_1, \ldots, y_n)$$

there is an argument (the same $y_s$) standing in the place $s$ occupied by $x$ in $(*)$.

Since in equalities $(*)$ and $(**)$ all arguments are independent, these equalities will hold also after any transformation of $\mathfrak{A}$ into $\mathfrak{A}$. Now apply in $(*)$ the transformation

$$x \to x, \ x_i \to x_1 \text{ if } x_i \neq x,$$

and in $(**)$ the transformation

$$y_{q_i+j_i} \to y_s \text{ (where } j_i \leqslant r_i), \ y_i \to y \text{ in other cases}.$$

We have

$$x = f(\underbrace{x_1, \ldots, x_1}_{q_1 \text{ times}}, \underbrace{x, \ldots, x}_{r_1 \text{ times}}, \underbrace{x_1, \ldots, x_1}_{q_1 \text{ times}}, \ldots),$$

$$y = f(\underbrace{y, \ldots, y}_{q_1 \text{ times}}, \underbrace{y_s, \ldots, y_s}_{r_1 \text{ times}}, \underbrace{y, \ldots, y}_{q_2 \text{ times}}, \ldots),$$

but this is impossible in view of $y \neq y_s$ and $x \neq x_1$. This completes the proof.

**2. Other independences.** Except for $C$-independence and independence, there are same other conditions given by Schmidt [5] (see also [3]). Namely, a subset $I$ of an abstract algebra is said to be *$S_i$-independent* $(i = 1, 2, 3)$ if it satisfies the following conditions:

$(S_1)$ For each $x \in I$, $C(\{x\}) \cap C(I \setminus \{x\}) = C(\emptyset)$ and $I \cap C(\emptyset) = \emptyset$.

$(S_2)$ For each two disjoint subsets $A, B$ of $I$, $C(A) \cap C(B) = C(\emptyset)$ and $I \cap C(\emptyset) = \emptyset$.

$(S_3)$ For each two subsets $A, B$ of $I$, $C(A) \cap C(B) = C(A \cap B)$ and $I \cap C(\emptyset) = \emptyset$.

EXAMPLE 2.1. Proposition 1.1 fails for $S_1$-independence. Indeed, let $\mathfrak{A} = (0, 1, 2, 3; f)$, where $f(0) = f(1) = f(2) = 2$ and $f(3) = 3$, and let $\mathfrak{B} = (a, b, c; f)$ be a homomorphic image of $\mathfrak{A}$ by the function $h$ defined as follows: $h(0) = a$, $h(1) = b$, $h(2) = h(3) = c$. It is easy to see that $\{a, b\}$ is $S_1$-independent in $\mathfrak{B}$, but $\{0, 1\}$ is not $S_1$-independent in $\mathfrak{A}$.

Let us observe that in Example 2.1 the set $\{a, b\}$ is independent even in $\mathfrak{B}$.

EXAMPLE 2.2. Proposition 1.2 fails for $S_1$-independence. It suffices to consider $\mathfrak{A} \times \mathfrak{B}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are such as in Example 2.1.

Moreover, Example 2.2 shows that a selector of an independent set fails to be $S_1$-independent in the product.

Proposition 1.3 can be generalized to $S_i$-independence as follows:

PROPOSITION 2.3. *Let $\mathfrak{A}$ be a given algebra, $\{a_0, \ldots, a_{n-1}\}$ be independent in $\mathfrak{A}$, and $\{b_0, b_1\}$ be $S_i$-independent in $\mathfrak{A}$. Then the set $I = \{(a_0, b_0), \ldots, (a_{n-1}, b_0), (a_{n-1}, b_1)\}$ is $S_i$-independent in $\mathfrak{A}^2$ $(i = 1, 2)$.*

Proof is analogous to that of Proposition 1.3.

We do not know whether Proposition 2.3 remains true for $i = 3$ (**P 701**). Independence has not this property:

EXAMPLE 2.4. Let $\mathfrak{P}^*$ be an upper Post algebra, that is $\mathfrak{P}^* = (0, 1; p^*)$, where $p^*(x, x, y) = p^*(x, y, x) = p^*(y, x, x) = x$. Consider an algebra $\mathfrak{R} = \mathfrak{P}^* \times \mathfrak{P}^*$. Then $\{0, 1\}$ is independent in $\mathfrak{P}^*$, but $\{(0, 0), (0, 1), (1, 1)\}$ is not independent in $\mathfrak{R}$.

We can show that Proposition 1.4 fails for $S_1$, $S_2$, and $S_3$. The additional assumption that projections of $P$ are independent is of no avail.

EXAMPLE 2.5. Let us consider the product $\mathfrak{A} \times \mathfrak{B}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are the same as in Example 2.1. Put $P = \{(0, a), 0, b)\}$. It is clear that $P$ satisfies the condition from Proposition 1.4 and the sets $\{0\}$ and $\{a, b\}$ are independent in $\mathfrak{A}$ and $\mathfrak{B}$, respectively. But $P$ is not $S_i$-independent in $\mathfrak{A} \times \mathfrak{B}$ for $i = 1, 2, 3$.

Note that if in Proposition 1.4 we assume that $\mathfrak{A}_t = \mathfrak{A}$, then the following holds:

PROPOSITION 2.6. *Let $\mathfrak{A} = \mathfrak{B}^T$ and let $P$ be a subset of $A$ such that, for each $t \in T$, its projection $\pi_t(P)$ is an independent subset of $B$. Then each subset $Q \subset P$ satisfying the condition*

$$\forall_{p \in Q} \exists_{t \in T} \forall_{q \in Q} [p \neq q \Rightarrow p(t) \neq q(t)]$$

*is $S_1$-independent in $\mathfrak{A}$.*

We do not know whether Proposition 2.6 remains true for $S_2$ or $S_3$ (**P 702**).

**3. EIS and products.** In this section we give an example to show that products do not preserve condition of exchange of independent sets denoted EIS (for the definition and properties of EIS see, e.g. [2] or [4]).

EXAMPLE 3.1. Let $K$ be the class of all semigroups satisfying the equations: $x^2 = y^2$, $xyz = x^2$, $xy = yx$. It is clear that

(i) It $\mathfrak{A}$ is free over $K$ and has two $K$-free generators, then EIS holds in $\mathfrak{A}$.

(ii) If $\mathfrak{B}$ is free over $K$ and has three $K$-free generators, then EIS does not hold in $\mathfrak{B}$.

(iii) $\mathfrak{B}$ can be embedded into $\mathfrak{A}^2$.

From (i), (ii), and (iii) we infer that $K$-free generators of $\mathfrak{B}$ form independent sets in $\mathfrak{A}^2$ and so that EIS is not invariant under products.

Remark. B. Jónsson has communicated in a letter to E. Marczewski that from [1] it follows

THEOREM 3.2 (Jónsson). *An algebra $\mathfrak{A}$ has EIS if the class of free algebras over $HSP(\mathfrak{A})$ (it is, the smallest equational class containing $\mathfrak{A}$) has amalgamation property.*

Our example shows that the converse thèorem is false (this fact was known also to S. Fajtlowicz). Indeed (under notation of Example 3.1), $\mathfrak{A}$ has EIS but the class of all free algebras over $HSP(\mathfrak{A})$ has not amalgamation property, since $\mathfrak{B}$ has not EIS and $HSP(\mathfrak{A}) = HSP(\mathfrak{B})$.

## REFERENCES

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