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POLYNOMIAL INEQUALITIES ON GENERAL SUBSETS OF R^N

BY

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Introduction. The classical Markov inequality for polynomials

$$\sup_{|x|\leqslant 1} |P'(x)| \leqslant (\deg P)^2 \sup_{|x|\leqslant 1} |P(x)| \tag{1}$$

is closely related to the polynomial division inequality

$$\sup_{|x|\leqslant 1} |P| \leqslant (\deg P+1)^2 \sup_{|x|\leqslant 1} |(x-a)P(x)| \quad (a \in \boldsymbol{R}). \tag{2}$$

Actually it is not difficult to show that (2) can be established using (1) and that (2) implies immediately

$$\sup_{|x|\leqslant 1} |P'(x)| \leqslant 4(\deg P)^2 \sup_{|x|\leqslant 1} |P(x)|.$$

The aim of the first part of this paper is to extend this kind of results to subsets of $\boldsymbol{R}^N$: we prove that if Ω is a bounded subset of $\boldsymbol{R}^N$ and if we have a Markov inequality in $L^p(\Omega)$, then we have a polynomial division inequality in $L^p(\Omega)$. We know that Markov inequality holds in $L^p(\Omega)$ if Ω is a bounded Lipschitz subset of $\boldsymbol{R}^N$ (see [1]), but the most general known subsets of $\boldsymbol{R}^N$ for which we have a Markov inequality in L^∞ are uniformly polynomially cuspidal sets, recently introduced by Pawłucki and Pleśniak [3]. In the second part, we prove that we also have a Markov inequality in L^p for these subsets, and therefore a polynomial division inequality in L^p, and we give an application to characterization of C^∞-functions.

0. Notation and definitions. Throughout this paper we use the classical multivariate notation:

N is the set of non-negative integers;

for $a = (a_1, \ldots, a_N) \in N^N$,

$$|a| = \sum_{i=1}^{N} a_i;$$

for a and b in N^N,

$$\binom{a}{b} = \prod_{i=1}^{N} \binom{a_i}{b_i}.$$

We denote by $B(c, r)$ the N-dimensional ball with center $c \in \boldsymbol{R}^N$ and radius $r > 0$, by H_n the set of polynomials in N variables of degree at most n, and by $L^p(\Omega)$ the space of measurable functions satisfying

$$\|f\|_{p,\Omega} = \Big[\int_\Omega |f(x)|^p\, dx\Big]^{1/p} < \infty \qquad (1 \leqslant p < \infty),$$

$$\|f\|_{\infty,\Omega} = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| < \infty \qquad (p = \infty).$$

Let $\Omega \subset \boldsymbol{R}^N$ and f be a real-valued function defined on $\bar{\Omega}$. We say that f *vanishes* on $\bar{\Omega}$ at order at most d if for any $x \in \bar{\Omega}$ there exists $\alpha \in N^N$ such that $|\alpha| \leqslant d$ and $f^{(\alpha)}(x) \neq 0$.

1. Markov inequality implies polynomial division inequality.

1.1. *Statement of the result.* In this part, Ω is a bounded subset of $\boldsymbol{R}^N$, $p \in [1, \infty]$, and we assume that a Markov inequality holds in $L^p(\Omega)$, i.e.:

(MI) There exist two positive constants C and $r \geqslant 1$ such that, for every $P \in H_n$,

$$\|D_i P\|_{p,\Omega} \leqslant Cn^r \|P\|_{p,\Omega} \qquad (i = 1, \ldots, N).$$

We shall prove the following result:

THEOREM 1 (with previous notation and under assumption (MI)). *Let $f \in C^s(\boldsymbol{R}^N)$ vanish on $\bar{\Omega}$ at order at most d with $rd < s$. Then there exists a positive constant $C(f, \Omega)$ such that for every $P \in H_n$ we have*

$$\|P\|_{p,\Omega} \leqslant C(f, \Omega)\, n^{rd} \|Pf\|_{p,\Omega} \qquad (n > 0).$$

1.2. *Basic results about Markov inequality.* First we recall the standard Markov inequality on an interval ([2], pp. 133 and 141): let $a < b < c < d$ and P be a polynomial of degree at most n; we have

$$\|P'\|_{\infty,[b,c]} \leqslant C(b, c)\, n^2 \|P\|_{\infty,[b,c]},$$

$$\|P'\|_{\infty,[b,c]} \leqslant C(a, b, c, d)\, n \|P\|_{\infty,[a,d]}.$$

We shall use an immediate extension of the second estimate to rectangular parallelepipeds:

LEMMA 1. *Let Π_1 and Π_2 be open rectangular parallelepipeds. Assume $\bar{\Pi}_1 \subset \Pi_2$. Then there exists a constant $C_1(\Pi_1, \Pi_2)$ such that for any $P \in H_n$ we have*

$$\|P^{(\alpha)}\|_{\infty,\Pi_1} \leqslant C_1(\Pi_1, \Pi_2)\, n \|P\|_{\infty,\Pi_2} \qquad (|\alpha| = 1).$$

1.3. *Approximation of a function and its derivatives on a subset of $\boldsymbol{R}^N$.* A fundamental tool is the following

LEMMA 2 ([4], Theorem 4.5, p. 167). *Let Π be a rectangular parallelepiped and $f \in C^s(\bar{\Pi})$. Then there exists a positive constant $C(\Pi)$ such that, for any $n \in \boldsymbol{N}^*$, a polynomial $R_n \in H_n$ can be found satisfying*

$$\|f - R_n\|_{\infty,\Pi} \leqslant C(\Pi) n^{-s} \operatorname*{Max}_{|\alpha|=s} \{\|f^{(\alpha)}\|_{\infty,\Pi}\}.$$

COROLLARY 1. *Let Ω be a bounded subset of $\boldsymbol{R}^N$ and $f \in C^s(\boldsymbol{R}^N)$. Then there exists a constant $C_2(f, \Omega)$ such that for any $n \in \boldsymbol{N}^*$ one can find $R_n \in H_n$ satisfying*

$$\|f^{(\gamma)} - R_n^{(\gamma)}\|_{\infty,\Omega} \leqslant C_2(f, \Omega) n^{|\gamma|-s} \qquad (|\gamma| < s).$$

Proof. We choose open rectangular parallelepipeds $\Pi_0, \ldots, \Pi_{s-1}$ such that $\bar{\Omega} \subset \Pi_0$, $\bar{\Pi}_i \subset \Pi_{i+1}$ $(i = 0, 1, \ldots, s-2)$. Then, by Lemma 2, for every $n \in \boldsymbol{N}^*$ one can find $R_n \in H_n$ such that

$$\|f - R_n\|_{\infty,\Pi_{s-1}} \leqslant C(\Pi_{s-1}) n^{-s} \operatorname*{Max}_{|\alpha|=s} \{\|f^{(\alpha)}\|_{\infty,\Pi_{s-1}}\} = C_1 n^{-s},$$

all constants $C_1, C_2, \ldots$ depending on f and Ω. For every $x \in \Pi_{s-1}$,

$$f(x) - R_n(x) = \sum_{k=0}^{\infty} (R_{2^{k+1}n} - R_{2^k n})(x).$$

We now prove that for $|\gamma| = 1$ the series

$$\sum_{k=0}^{\infty} (R_{2^{k+1}n} - R_{2^k n})^{(\gamma)}$$

is absolutely convergent in $L^\infty(\Pi_{s-2})$. Using Lemma 1 gives

$$\begin{aligned}
&\|(R_{2^{k+1}n} - R_{2^k n})^{(\gamma)}\|_{\infty,\Pi_{s-2}} \\
&\quad \leqslant C_1(\Pi_{s-1}, \Pi_{s-2}) \cdot 2^{k+1} n \|R_{2^{k+1}n} - R_{2^k n}\|_{\infty,\Pi_{s-1}} \\
&\quad \leqslant C_1(\Pi_{s-1}, \Pi_{s-2}) \cdot 2^{k+1} n (\|R_{2^{k+1}n} - f\|_{\infty,\Pi_{s-1}} + \|f - R_{2^k n}\|_{\infty,\Pi_{s-1}}) \\
&\quad \leqslant C_2 \cdot 2^{k+1} n (2^k n)^{-s},
\end{aligned}$$

and then

$$\sum_{k=0}^{\infty} \|(R_{2^{k+1}n} - R_{2^k n})^{(\gamma)}\|_{\infty,\Pi_{s-2}} \leqslant C_3 n^{1-s},$$

whence

$$\|(f - R_n)^{(\gamma)}\|_{\infty,\Pi_{s-2}} \leqslant C_3 n^{1-s}.$$

Iterating this process leads to

$$\|(f - R_n)^{(\gamma)}\|_{\infty,\Pi_{s-|\gamma|-1}} \leqslant C_4 n^{|\gamma|-s} \qquad (|\gamma| = 1, \ldots, s-1),$$

which proves the corollary due to the fact that

$$\|(f-R_n)^{(\gamma)}\|_{\infty,\Omega} \leqslant \|(f-R_n)^{(\gamma)}\|_{\infty,\Pi_i} \quad (i = 0, \ldots, s-1).$$

1.4. *Polynomial inequalities.*

LEMMA 3 (with previous notation and under assumption (MI)). *Let $d \in N$. Then for every $A \subset \bar{\Omega}$ and $R \in H_j$ satisfying the condition*

$$\{\text{there exists } \alpha \in N^N \text{ such that } |\alpha| \leqslant d \text{ and } |R^{(\alpha)}(x)| \geqslant m_0 > 0 \ (x \in A)\}$$

one can find a constant C_d such that for any $\varepsilon > 0$ and every $P \in H_n$ we have

$$\|P\|_{p,A} \leqslant C_d m_0^{-1} \varepsilon^{-d} (n+j)^{rd} \|PR\|_{p,\Omega} + \varepsilon \|P\|_{p,\Omega}. \tag{3}$$

Proof. We proceed by induction on the length of α.

(1) Assume $|\alpha| = 0$. Then, for $x \in A$,

$$|R(x)| \geqslant m_0 \quad \text{and} \quad |P(x)| \leqslant m_0^{-1} |P(x) R(x)|.$$

Therefore

$$\|P\|_{p,A} \leqslant m_0^{-1} \|PR\|_{p,A} \leqslant m_0^{-1} \|PR\|_{p,\Omega} + \varepsilon \|P\|_{p,\Omega}.$$

(2) Let $d_0 \leqslant d$. We assume that (3) holds when $|\alpha| < d_0$. Let α be such that $|\alpha| = d_0$ and prove that (3) is still valid. Let

$$D = \{k \in N^N\colon\ 0 < |k|,\ 0 \leqslant k_i \leqslant a_i \ (i = 1, \ldots, N)\}.$$

From the Leibniz formula, if $x \in A$,

$$|P(x)| \leqslant m_0^{-1} \left\{ |(PR)^{(\alpha)}(x)| + \sum_{k \in D} \binom{\alpha}{k} |R^{(\alpha-k)}(x)|\, |P^{(k)}(x)| \right\}.$$

We set

$$L = \binom{N+d_0}{d_0}$$

(there are at most $L-1$ terms in the sum $\sum$) and

$$B_0 = \left\{ x \in A\colon |R^{(\alpha-k)}(x)| \leqslant \binom{\alpha}{k}^{-1} m_0 \varepsilon (Cn^r)^{-|k|} L^{-2},\ k \in D \right\}.$$

If $x \in B_0$, then

$$|P(x)| \leqslant m_0^{-1} |(PR)^{(\alpha)}(x)| + \varepsilon L^{-2} \sum_{k \in D} (Cn^r)^{-|k|} |P^{(k)}(x)|.$$

This yields

$$\begin{aligned} \|P\|_{p,B_0} &\leqslant m_0^{-1} \|(PR)^{(\alpha)}\|_{p,B_0} + \varepsilon L^{-2} \sum_{k \in D} (Cn^r)^{-|k|} \|P^{(k)}\|_{p,B_0} \\ &\leqslant m_0^{-1} \|(PR)^{(\alpha)}\|_{p,\Omega} + \varepsilon L^{-2} \sum_{k \in D} (Cn^r)^{-|k|} \|P^{(k)}\|_{p,\Omega}, \end{aligned}$$

and using (MI) we obtain

$$\|P\|_{p,B_0} \leqslant m_0^{-1} C^{r|\alpha|}(n+j)^{r|\alpha|}\|PR\|_{p,\Omega}+\varepsilon L^{-1}\|P\|_{p,\Omega}$$

(there are less than L terms in $\sum$). For every $x \in A\setminus B_0$ there exists $k \in D$ such that

$$|R^{(\alpha-k)}(x)| > \binom{\alpha}{k}^{-1} m_0\,\varepsilon (Cn^r)^{-|k|} L^{-2}. \tag{4}$$

Then one can share $A\setminus B_0$ into at most $L-1$ disjoint subsets B_i such that for every $x \in B_i$ there exists an index k for which (4) holds true. Now, by induction, since $|k|>0$, on each B_i, replacing ε by ε/L, we have

$$\|P\|_{p,B_i} \leqslant (Cn^r)^{|k|} L^2 \binom{\alpha}{k}(m_0\,\varepsilon)^{-1} C_{d_0-1}(L/\varepsilon)^{d_0-1} \times (n+j)^{r(d_0-|k|)}\|PR\|_{p,\Omega}+(\varepsilon/L)\|P\|_{p,\Omega}.$$

Now, using the inequality

$$\|P\|_{p,A} \leqslant \|P\|_{p,B_0}+\sum_i \|P\|_{p,B_i}$$

yields

$$\|P\|_{p,A} \leqslant m_0^{-1} C_{d_0}\varepsilon^{-d_0}(n+j)^{rd_0}\|PR\|_{p,\Omega}+\varepsilon\|P\|_{p,\Omega}$$

with

$$C_{d_0} = C^{rd_0}+L^{d_0+1} C_{d_0-1}\sum_{k\in D}\binom{\alpha}{k}C^{|k|},$$

which proves the induction step.

1.5. Proof of Theorem 1. For every $a \in \bar{\Omega}$ there exists $\alpha \in N^N$ such that $|\alpha| \leqslant d$ and $f^{(\alpha)}(a) \neq 0$. Let V_a be a neighborhood of a such that, for any $x \in V_a \cap \bar{\Omega}$,

$$|f^{(\alpha)}(x)| > \tfrac{1}{2}|f^{(\alpha)}(a)|.$$

Since $\bar{\Omega}$ is compact, it can be covered by a finite number of such neighborhoods denoted by $V_{a_1}, \ldots, V_{a_k}$. Now, for $i \in \{1, \ldots, k\}$ there exists $\alpha^i \in N^N$ satisfying $|\alpha^i| \leqslant d$ and such that for $x \in V_{a_i} \cap \bar{\Omega}$ we have

$$|f^{(\alpha^i)}(x)| > \tfrac{1}{2}|f^{(\alpha^i)}(a_i)|. \tag{5}$$

On the other hand, $d < s$ and there exists n_0 such that for $n > n_0$ we have

$$C_2(f, \Omega)\, n^{d-s} \leqslant \tfrac{1}{4}\operatorname*{Min}_i \{|f^{(\alpha^i)}(a_i)|\},$$

where $C_2(f, \Omega)$ is the constant of Corollary 1. Then for $n > n_0$ the polynomial R_n of Lemma 2 satisfies

$$|f^{(\alpha^i)}(x)-R_n^{(\alpha^i)}(x)| \leqslant \tfrac{1}{4}|f^{(\alpha^i)}(a_i)| \quad (x \in V_{a_i} \cap \bar{\Omega},\ i = 1, \ldots, k),$$

and from (5) we get

$$(6) \qquad |R_n^{(\alpha^i)}(x)| \geqslant \tfrac{1}{4}|f^{(\alpha^i)}(a_i)| \quad (x \in V_{a_i} \cap \bar{\Omega},\ i = 1, \ldots, k).$$

To complete the proof we need the following

LEMMA 4. *There exists a constant $C_3(f, \Omega)$ such that if $R_n \in H_n$ is the polynomial of Lemma 2, then for any $P \in H_n$ $(n > 0)$ we have*

$$\|P\|_{p,\Omega} \leqslant C_3(f, \Omega)\, n^{2d} \|PR_n\|_{p,\Omega}.$$

Proof. Applying (6) and Lemma 3 on every $V_{a_i} \cap \bar{\Omega}$ with $\varepsilon = 1/2k$ yields

$$\|P\|_{p,V_{a_i}\cap\bar{\Omega}} \leqslant \left(4/|f^{(\alpha^i)}(a_i)|\right) C_d (2k)^d (2n)^{rd} \|PR_n\|_{p,\Omega} + \frac{1}{2k}\|P\|_{p,\Omega}.$$

Therefore

$$\|P\|_{p,\Omega} \leqslant \sum_{i=1}^{k} \|P\|_{p,V_{a_i}\cap\Omega}$$
$$\leqslant 4C_d (2k)^d \Big\{\sum_{i=1}^{k} |f^{(\alpha^i)}(a_i)|^{-1}\Big\} (2n)^{rd} \|PR_n\|_{p,\Omega} + \tfrac{1}{2}\|P\|_{p,\Omega},$$

and then

$$\|P\|_{p,\Omega} \leqslant C_3(f, \Omega)\, n^{rd} \|PR_n\|_{p,\Omega}.$$

To complete the proof of Theorem 1 observe that from Lemma 4 we get

$$\|P\|_{p,\Omega} \leqslant C_3(f, \Omega)\, n^{rd} \{\|P(R_n - f)\|_{p,\Omega} + \|Pf\|_{p,\Omega}\}$$
$$\leqslant C_3(f, \Omega)\, n^{rd} \{\|P\|_{p,\Omega} \|R_n - f\|_{\infty,\Omega} + \|Pf\|_{p,\Omega}\},$$

and using Corollary 1 we obtain

$$\|P\|_{p,\Omega} \leqslant C_4(f, \Omega)\, n^{rd-s} \|P\|_{p,\Omega} + C_3(f, \Omega)\, n^{rd} \|Pf\|_{p,\Omega}.$$

Let n_0 be such that $C_4(f, \Omega)\, n_0^{rd-s} < \frac{1}{2}$ (which is possible since $rd < s$). Then for $n > n_0$ we get

$$(7) \qquad \|P\|_{p,\Omega} \leqslant 2C_3(f, \Omega)\, n^{rd} \|Pf\|_{p,\Omega},$$

which proves the theorem for $n > n_0$. If $n \leqslant n_0$, (7) holds true due to the fact that a linear mapping from a finite dimensional space into another is always continuous.

2. Markov inequality holds in L^p-spaces on sets with polynomial cusps.

2.1. *Uniformly polynomially cuspidal sets.* Uniformly polynomially cuspidal sets (UPC sets) have been introduced by Pawłucki and Pleśniak in [3].

DEFINITION. A subset Ω of R^N is *uniformly polynomially cuspidal* if there exist positive constants M, m and a positive integer k such that for any $x \in \bar{\Omega}$ one can choose a polynomial map h_x: $R \to R^N$ of degree at most k such that

(i) $h_x((0, 1]) \subset \Omega$, $h_x(0) = x$;

(ii) for any $x \in \bar{\Omega}$ and any $t \in [0, 1]$,

$$\operatorname{dist}(h_x(t), R^N \setminus \Omega) \geqslant Mt^m.$$

UPC sets are a large class of sets. Examples: Lipschitz sets, bounded convex sets with non-void interior, fat subanalytic subsets of R^N are UPC sets (see [3] for details).

In the following we assume $m \geqslant 1$, which is not restrictive. Pawłucki and Pleśniak proved ([3], p. 469) that Markov inequality (MI) holds in L^∞ on UPC bounded subsets of R^N (and C^N). The proof of this result shows that $r = 2+2m$.

2.2. *Comparison of uniform and L^p-norms of polynomials.*

THEOREM 2. *Let Ω be a bounded UPC subset of R^N. For any polynomial $P \in H_n$ and any $p \geqslant 1$ we have*

$$\|P\|_{\infty,\Omega} \leqslant 2[K(p+1)\ldots(p+N)]^{1/p} n^{(rN)/p} \|P\|_{p,\Omega}$$

with

$$K = [(4k^2)^m \operatorname{Max}(M^{-1}, 2CN^{1/2})]^N/[V_N N!],$$

where M, k, m are the constants of the definition, C, r the constants of (MI), *and V_N the volume of the unit ball in R^N.*

Proof. Let $p \geqslant 1$ and $P \in H_n$. Let $a \in \bar{\Omega}$ be such that $\|P\|_{\infty,\Omega} = |P(a)|$. By the definition of a UPC set, for any $t \in [0, 1]$ we have

$$|P(h_a(t)) - P(a)| = |P(h_a(t)) - P(h_a(0))| = |Q(t) - Q(0)|,$$

where Q is a polynomial of a single variable of degree at most nk. Then, due to (1),

$$|P(h_a(t)) - P(a)| \leqslant t \operatorname*{Sup}_{[0,1]} |Q'| \leqslant 2tn^2k^2 \operatorname*{Sup}_{[0,1]} |Q| \leqslant 2tn^2k^2 \|P\|_{\infty,\Omega}.$$

Let $B = B(h_a(t), Mt^m)$. For any $x \in B$

$$P(x) - P(h_a(t)) = \overrightarrow{(x - h_a(t))} \cdot \overrightarrow{\operatorname{grad} P(y)}$$

for some y in the segment with ends at x and $h(t)$. Then using (MI) for $p = \infty$ we get

$$|P(x) - P(h_a(t))| \leqslant \operatorname{dist}(x, h_a(t)) CN^{1/2} n^r \|P\|_{\infty,\Omega},$$

and therefore

$$|P(x) - P(a)| \leqslant |P(a)| \left(2tn^2k^2 + CN^{1/2} n^r \operatorname{dist}(x, h_a(t))\right),$$

whence

(8) $$|P(a)|\left(1-2tn^2k^2-CN^{1/2}n^r \operatorname{dist}(x, h_a(t))\right) \leqslant |P(x)| \quad (x \in B).$$

We now choose

(9) $$t = (1/(4k^2))[\operatorname{Max}(1, 2CMN^{1/2})]^{-1/m} n^{-r/m}.$$

Since $r = 2+2m$, we have

$$t < 1/(4n^2k^2) \quad \text{and} \quad 1-2tn^2k^2 > 1/2.$$

Moreover,

$$CN^{1/2}n^r \operatorname{dist}(x, h_a(t)) \leqslant CN^{1/2}n^r Mt^m \leqslant 1/(2(4k^2)^m) < 1/2.$$

Then, by (8),

$$|P(a)|\left(1-2CN^{1/2}n^r \operatorname{dist}(x, h_a(t))\right) \leqslant 2|P(x)| \quad (x \in B).$$

We have

$$2CMN^{1/2} \leqslant (4k^2)^m \operatorname{Max}(1, 2CMN^{1/2})$$

and

$$2CN^{1/2}n^r \leqslant (4k^2)^m \operatorname{Max}(M^{-1}, 2CN^{1/2})n^r = 1/(Mt^m),$$

which implies

$$|P(a)|\left(1-\operatorname{dist}(x, h_a(t))/(Mt^m)\right) \leqslant 2|P(x)| \quad (x \in B).$$

Integrating the p-th power of both sides of the last estimate on B we have

$$\|P\|_{\infty,\Omega}^p N V_N \int_0^{Mt^m} (1-R/(Mt^m))^p R^{N-1}\, dR \leqslant 2^p \|P\|_{p,\Omega}^p,$$

which is the same as

$$\|P\|_{\infty,\Omega}^p N V_N (Mt^m)^N (N-1)!/[(p+1)\ldots(p+N)] \leqslant 2^p \|P\|_{p,\Omega}^p.$$

Replacing t by its value (9) leads to the inequality of Theorem 2.

2.3. *Comparison of different L^p-norms for polynomials.*

COROLLARY 2. *Let $q \geqslant p \geqslant 1$ and Ω be a bounded UPC subset of $\mathbf{R}^N$. Then for any $P \in H_n$ we have*

$$\|P\|_{q,\Omega} \leqslant 2[K(p+1)\ldots(p+N)]^{(1/p)-(1/q)} n^{rN((1/p)-(1/q))} \|P\|_{p,\Omega},$$

where K is the constant of Theorem 2, and r the constant of (MI) *with $p = \infty$.*

This corollary is a generalization of a similar result proved by Timan when Ω is a segment of the real line (see inequality (36) in [5], p. 236).

Proof. We have

$$\|P\|_{q,\Omega}^q = \int_\Omega |P(x)|^{q-p} |P(x)|^p\, dx \leqslant \|P\|_{\infty,\Omega}^{q-p} \|P\|_{p,\Omega}^p$$

and, using Theorem 2, we get

$$\|P\|_{q,\Omega}^{q} \leqslant 2^{q-p}[K(p+1)\dots(p+N)]^{(q/p)-1}\, n^{rN((q/p)-1)}\|P\|_{p,\Omega}^{q}.$$

Rising the two sides of the last inequality to the power $1/q$ gives the required inequality.

2.4. *Division inequality and Markov inequality in L^p-spaces.*

COROLLARY 3. *Let $p \geqslant 1$ and Ω be a bounded UPC subset of $\boldsymbol{R}^N$. Then for any $P \in H_n$ we have*

$$\|P^{(\alpha)}\|_{p,\Omega} \leqslant 2[K'(p+1)\dots(p+N)]^{1/p}\, n^{r(1+(N/p))}\|P\|_{p,\Omega} \qquad (|\alpha| = 1),$$

where K' is a constant depending only on Ω.

Proof. Clearly, using (MI) and Theorem 2 gives

$$\|P^{(\alpha)}\|_{p,\Omega} \leqslant [\mathrm{Mes}(\Omega)]^{1/p}\|P^{(\alpha)}\|_{\infty,\Omega} \leqslant [\mathrm{Mes}(\Omega)]^{1/p}\, Cn^r\|P\|_{\infty,\Omega}$$
$$\leqslant 2[\mathrm{Mes}(\Omega)K(p+1)\dots(p+N)]^{1/p}\, n^{r+rN/p}\|P\|_{p,\Omega}.$$

Theorem 1 yields

COROLLARY 4. *Let $p \geqslant 1$, Ω be a bounded UPC subset of $\boldsymbol{R}^N$ and $f \in C^s(\Omega)$ vanishing on Ω at order at most d with $rd(1+(N/p)) < s$. Then there exists a constant K'' depending on Ω, f and p such that for any $P \in H_n$*

$$\|P\|_{p,\Omega} \leqslant K''\, n^{rd(1+(N/p))}\|Pf\|_{p,\Omega}.$$

2.5. *Characterization of C^∞-functions.* A sequence (u_n) is said to be *rapidly decreasing* if and only if, for any $j > 0$, $\mathrm{Lim}(n^j u_n) = 0$.

LEMMA 5. *Let Ω be a bounded UPC subset of $\boldsymbol{R}^N$, and f a real-valued function defined on Ω. Then $(\mathrm{dist}_{L^p(\Omega)}(f, H_n))_{n\in N}$ is rapidly decreasing if and only if $(\mathrm{dist}_{L^\infty(\Omega)}(f, H_n))_{n\in N}$ is rapidly decreasing.*

Proof. Assume $(\mathrm{dist}_{L^p(\Omega)}(f, H_n))_{n\in N}$ is rapidly decreasing. For every n, let P_n be such that

$$\|f-P_n\|_{p,\Omega} = \mathrm{dist}_{L^p(\Omega)}(f, H_n).$$

Clearly, in $L^p(\Omega)$,

$$f = P_0 + \sum_{i=0}^{\infty}(P_{i+1}-P_i) \quad \text{and} \quad f-P_n = \sum_{i=n}^{\infty}(P_{i+1}-P_i).$$

Now, by Theorem 2, for any $i \in \boldsymbol{N}$:

$$\|P_{i+1}-P_i\|_{\infty,\Omega} \leqslant C_6(i+1)^{rN/p}\|P_{i+1}-P_i\|_{p,\Omega}$$
$$\leqslant C_6(i+1)^{rN/p}\, 2\,\mathrm{dist}_{L^p(\Omega)}(f, H_i),$$

whence we conclude that $P_0 + \sum_{i=0}^{\infty}(P_{i+1}-P_i)$ converges to f in $L^\infty(\Omega)$. Moreover,

$$\operatorname{dist}_{L^\infty(\Omega)}(f, H_n) = \|f-P_n\|_{\infty,\Omega} \leqslant \sum_{i=n}^{\infty} \|P_{i+1}-P_i\|_{\infty,\Omega}$$

$$\leqslant \sum_{i=n}^{\infty} 2C_6(i+1)^{rN/p} \operatorname{dist}_{L^p(\Omega)}(f, H_i),$$

which guarantees that $(\operatorname{dist}_{L^\infty(\Omega)}(f, H_n))_{n\in N}$ is rapidly decreasing.

The converse part of the lemma is proved in the same lines except that we need not Markov inequality since for any measurable function f we have

$$\|f\|_{p,\Omega} \leqslant (\operatorname{Mes}(\Omega))^{1/p} \|f\|_{\infty,\Omega}.$$

PROPOSITION 1. *Let $p \geqslant 1$ and Ω be a bounded UPC subset of $\boldsymbol{R}^N$. Then a real-valued function f defined on Ω is the restriction to Ω of a C^∞-function $\bar{f}$ in $\boldsymbol{R}^N$ if and only if $(\operatorname{dist}_{L^p(\Omega)}(f, H_n))_{n\in N}$ is a rapidly decreasing sequence.*

Proof. This is an immediate corollary to Lemma 5 and a theorem by Pawłucki and Pleśniak ([3], Theorem 5.1, p. 472), establishing Proposition 1 in the case $p = +\infty$.

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