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ON A QUESTION OF FREMLIN CONCERNING ORDER BOUNDED AND REGULAR OPERATORS

BY

JU. A. ABRAMOVIČ AND V. A. GEJLER (LENINGRAD)

For vector lattices E and F (always assumed Archimedean) we denote by $L^b(E, F)$ the space of all order bounded operators (1) from E into F, and by $L^r(E, F)$ the subspace of $L^b(E, F)$ consisting of all regular operators. The cone of positive elements in E will be denoted by E_+.

If F is a K-space (i.e., Dedekind complete vector lattice), then the well-known Kantorovich theorem asserts that $L^b(E, F) = L^r(E, F)$ and $L^r(E, F)$ is a K-space for every vector lattice E (see, e.g., [3]). On the other hand, it is also known that generally $L^b(E, F) \neq L^r(E, F)$ (see, e.g., [1]). In connection with these two facts D. H. Fremlin suggested (in a letter to the first-named author) the following question:

(Q1) Does there exist a vector lattice F such that F is not a K-space but for every vector lattice E the equality $L^b(E, F) = L^r(E, F)$ holds?

Proposition 2 below gives a positive answer to this question. In connection with this result it is natural to ask a question stronger than (Q1), namely:

(Q2) Does there exist a vector lattice F such that F is not a K-space but for every vector lattice E the equality $L^b(E, F) = L^r(E, F)$ holds and, additionally, $L^r(E, F)$ is a vector lattice?

The Theorem in the sequel gives a negative answer to (Q2).

For a vector lattice F we denote by $\hat{F}$ its Dedekind completion and we identify F with its canonical image in $\hat{F}$.

PROPOSITION 1. *Let F be a vector lattice such that there exists an operator $R : \hat{F} \to F$ with the following property:*

(*) *for every $\hat{x} \in \hat{F}_+$, $R\hat{x} \geqslant \hat{x}$.*

Then the equality $L^r(E, F) = L^b(E, F)$ holds for every vector lattice E.

(1) We recall that an operator $T : E \to F$ is said to be *order bounded* (respectively, *regular*) if T maps order bounded subsets from E into order bounded subsets in F (respectively, if T is a difference of two positive operators).

Proof. Let $T \in L^b(E, F)$. Thus, moreover, $T \in L^b(E, \hat{F})$ and, according to the above-mentioned Kantorovich theorem, there exists an operator $T_+ \in L^b(E, \hat{F})$ (a positive part of T). We put $T_1 = RT_+$. It is clear that $T_1 \in L^b(E, F)$ and $T_1 \geqslant 0$. Further, for every $x \in E_+$ we have

$$Tx \leqslant T_+ x \leqslant RT_+ x = T_1 x,$$

and hence $T \in L^r(E, F)$.

PROPOSITION 2. *Let Q be an extremally disconnected Hausdorff compact space with the following property*:

There exist two distinct non-isolated points $q_1, q_2 \in Q$ and a homeomorphism $f\colon Q \to Q$ such that $f(q_1) = q_2$ and $f(q_2) = q_1$.

Then in the vector lattice $C(Q)$ of all continuous functions on Q there exists a vector sublattice F which is not a K-space but has property $(*)$.

Proof. Let us put $F = \{y \in C(Q)\colon y(q_1) = y(q_2)\}$. It is clear that $\hat{F} = C(Q) \neq F$. Hence F is not a K-space. Now we define an operator $R\colon \hat{F} \to F$ by setting

$$(R\hat{y})(q) = \hat{y}(q) + \hat{y}\big(f(q)\big), \qquad \hat{y} \in C(Q), \ q \in Q.$$

It is obvious that the image of R belongs to F and for every $\hat{y} \in C(Q)_+$ we have $R\hat{y} \geqslant \hat{y}$.

Remark. Of course, there exist plenty of extremally disconnected Hausdorff compact spaces Q with the property formulated in Proposition 2. For example, the space βN, the Čech-Stone compactification of the countable discrete space N, has this property.

For any set I we denote by $l^\infty(I)$ the K-space of all bounded functions on I. For $i \in I$ let e_i be the characteristic functions of the one-point set $\{i\}$, i.e., $e_i(j) = \delta_{ij}$ $(j \in I)$, and let 1 be the constant function on I which takes on I the value 1. We denote by $l_0^\infty(I)$ the vector subspace of $l^\infty(I)$ which is generated by the set $\{e_i : i \in I\} \cup \{1\}$. It is obvious that $l_0^\infty(I)$ is the sublattice of $l^\infty(I)$ consisting of all functions on I which are constants outside of finite subsets of I.

THEOREM. *Let F be a vector lattice. Then the following conditions are equivalent*:

(1) *F is a K-space*;

(2) *for every vector lattice E the equality $L^r(E, F) = L^b(E, F)$ holds and $L^r(E, F)$ is a K-space*;

(3) *for every vector lattice E the equality $L^r(E, F) = L^b(E, F)$ holds and $L^r(E, F)$ is a vector lattice*;

(4) *for every set I the space $L^r\big(l_0^\infty(I), F\big)$ is a vector lattice.*

Proof. We must prove only the implication (4) $\Rightarrow$ (1) because the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious and the implication (1) $\Rightarrow$ (2) is again the Kantorovich theorem.

First we shall prove that F is Dedekind σ-complete. Let (y_n) be an increasing sequence in F_+ such that $y_n \leqslant y$ for some $y \in F_+$. We set $E = l_0^\infty(N)$ and define an operator $T: E \to F$ by putting $T1 = 0$, $Te_1 = y_1$, and $Te_n = y_n - y_{n-1}$ $(n \geqslant 2)$. Let T_1 be another operator from E into F such that $T_1 1 = y$ and $T_1 e_n = Te_n$ $(n \geqslant 1)$. It is clear that $T_1 \geqslant 0$ and $T_1 \geqslant T$, so $T \in L^r(E, F)$. Since $L^r(E, F)$ is a vector lattice, there exists an operator $T_+ \in L^r(E, F)$. We show that

$$T_+ 1 = \sup_n y_n.$$

Indeed, for every $n = 1, 2, \ldots$ we have

$$T_+ 1 \geqslant T_+(e_1 + \ldots + e_n) \geqslant T(e_1 + \ldots + e_n) = y_n.$$

Thus $T_+ 1$ is an upper bound of (y_n).

Conversely, let $\bar{y}$ be some upper bound of (y_n). We define an operator $\bar{T}: E \to F$ by putting $\bar{T}1 = \bar{y}$ and $\bar{T}e_n = Te_n$ $(n \geqslant 1)$. Then $\bar{T} \geqslant 0$ and $\bar{T} \geqslant T$, thus $\bar{T} \geqslant T_+$. Consequently, $\bar{y} = \bar{T}1 \geqslant T_+ 1$, i.e., $T_+ 1$ is the least upper bound of (y_n), and hence F is Dedekind σ-complete.

Now the conclusion that F is a K-space will follow if we show that F is conditionally laterally complete, i.e., every order bounded family $(y_i)_{i \in I}$ of pairwise disjoint elements in F_+ has the least upper bound in F (see [2], Theorem 4). We put $E = l_0^\infty(I)$ and define an operator $T: E \to F$ by setting $T1 = 0$ and $Te_i = y_i$ $(i \in I)$. In the same way as above we can show that $T \in L^r(E, F)$ and $T_+ 1 = \sup\{y_i : i \in I\}$. The proof is completed.

Remark. We cannot replace (3) in the Theorem by the following condition:

There exists a (fixed) infinite-dimensional vector lattice E such that $L^r(E, F)$ is a vector lattice.

Indeed, the first-named author showed that the space $L^r(l_1, F)$ is a vector lattice for every Banach lattice F. The vector lattice F built in Proposition 2 is obviously a Banach lattice.

REFERENCES

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