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MODAL EXTENSIONS OF HEYTING ALGEBRAS

BY

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1. Introduction. In [10] the second-named author defines a *modal operator* on a lattice L to be a mapping $f: L \to L$ satisfying the following:

$x \leqslant f(x)$ (f is inflationary),

$f(f(x)) = f(x)$ (f is idempotent),

$f(x \wedge y) = f(x) \wedge f(y)$ (f is meet-preserving).

The theory of modal operators is shown to be particularly fruitful when L is assumed to be a Heyting algebra and has interesting applications to topology and sheafification. An account of this theory is forthcoming. In this paper we prove only those properties of modal operators that are necessary to meet our needs. The set $\mathfrak{M}(H)$ of all modal operators on a Heyting algebra H is called the *modal extension of H*. We show that if H is complete, then $\mathfrak{M}(H)$ is a complete Heyting algebra. Necessary and sufficient conditions on a complete Heyting algebra H are found in order that $\mathfrak{M}(H)$ is a Boolean algebra and used to show that if H satisfies the descending chain condition, then $\mathfrak{M}(H)$ is Boolean. Attention is then restricted to the study of modal operators on Post algebras. Specifically, a normal form is obtained for modal operators on a Post algebra H and used to show that $\mathfrak{M}(H)$ is a Boolean algebra. This fact is then applied to describe those P-algebras of type n whose modal extension is Boolean and the result is used to show that, for a three-valued Łukasiewicz algebra H, $\mathfrak{M}(H)$ is Boolean if and only if H has the smallest dense element.

2. Preliminaries. An algebra $\langle L; \wedge, {}^*, 0, 1\rangle$ is called a *pseudocomplemented semilattice* if and only if $\langle L; \wedge, 0, 1\rangle$ is a bounded semilattice such that for every $a \in L$ the element $a^* \in L$ is the *pseudocomplement of a*, that is $x \leqslant a^*$ if and only if $a \wedge x = 0$. If in any pseudocomplemented semilattice L we write

$$B(L) = \{x \in L; x = x^{**}\},$$

then $\langle B(L); \cup, \wedge, {}^*, 0, 1\rangle$ is a Boolean algebra, $a \cup b$ being defined by $a \cup b = (a^* \wedge b^*)^*$ for all $a, b \in B(L)$. The set

$$D(L) = \{x \in L;\, x^* = 0\}$$

is called a *dense filter* in L. For the standard rules of computation in pseudocomplemented (semi) lattices we refer to [1] or [6]. In any lattice whose dual is pseudocomplemented, we write a^+ for the *dual pseudocomplement of* a, that is $x \geqslant a^+$ if and only if $a \vee x = 1$.

A *Stone algebra* is a bounded, distributive, pseudocomplemented lattice in which $x^* \vee x^{**} = 1$ holds identically. A *double Stone algebra* is a Stone algebra whose dual is a Stone algebra.

A *three-valued Łukasiewicz* algebra is a double Stone algebra in which $x^* = y^*$ and $x^+ = y^+$ imply $x = y$. An account of three-valued Łukasiewicz algebras may be found in [11].

A *Brouwerian algebra* is an algebra $\langle H; \wedge, \vee, *\rangle$, where $\langle H; \wedge, \vee\rangle$ is a lattice and $*$ is a binary operation defined on H by $x \wedge y \leqslant z$ if and only if $x \leqslant y * z$. Every Brouwerian algebra is distributive and has the greatest element 1.

A *Heyting algebra* is a Brouwerian algebra with the least element 0. In a Heyting algebra H the element $x * 0$ is the pseudocomplement of x in H and if $a \in H$, then the interval $[a, 1] = \{x \in H;\, x \geqslant a\}$ is itself a Heyting algebra in which the pseudocomplement of x in $[a, 1]$ is $x * a$. For the standard rules of computation in Heyting algebras we refer to [1].

An *L-algebra* is a Heyting algebra in which $(x * y) \vee (y * x) = 1$ holds identically. Any L-algebra is a Stone algebra. The theory of L-algebras may be found in [8].

A *K-algebra of type* $n \geqslant 2$ is a Brouwerian algebra in which

$$\bigvee_{i=1}^{n} (x_i * x_{i+1}) = 1$$

holds identically. An *L-algebra of type* $n \geqslant 2$ is a K-algebra of type n having the least element 0. These algebras were introduced and developed in [7].

A *P-algebra* is an L-algebra whose dual is an L-algebra. Equivalently, a P-algebra is an L-algebra whose dual is a Stone algebra. For this and other characterizations of P-algebras we refer to [5].

A *P-algebra of type* $n \geqslant 2$ is an L-algebra of type n whose dual is an L-algebra.

A *Stone lattice of order* $n \geqslant 2$ is an L-algebra containing a chain $0 = e_0 < e_1 < \ldots < e_{n-1} = 1$ having the property that e_i is the smallest dense element in $[e_{i-1}, 1]$ $(1 \leqslant i \leqslant n-1)$. For the theory of Stone lattices of order n we refer the reader to [9].

A *P_2-algebra of order $n \geqslant 2$* is a Stone lattice of order n whose dual is a Stone algebra. For other characterizations and properties of P_2-algebras the reader is referred to [4].

Post algebras can be defined in various ways. By [5], a *Post algebra of order* $n \geqslant 2$ is a P_2-algebra of order n in which $e_{n-2}^{++} = 0$. Other definitions may be found in [3] and the references therein. It is well known that in any Post algebra H of order n, every element x can uniquely be represented in the form

$$x = \bigvee_{i=1}^{n-1} a_i \wedge e_i,$$

where a_i belongs to the centre $C(H)$ of H and $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_{n-1}$. Thus, we can define operators $D_i : H \to C(H)$ by $D_i(x) = a_i$ $(1 \leqslant i \leqslant n-1)$. Furthermore, Post algebras of order n are equationally definable as algebras of the form

$$\langle H;\ \wedge,\ \vee,\ *;\ D_1, D_2, \ldots, D_{n-1};\ e_0, e_1, \ldots, e_{n-1}\rangle$$

with 3 binary operations $(\wedge, \vee, *)$, $n-1$ unary operations $(D_1, D_2, \ldots \ldots, D_{n-1})$ and n nullary operations $(e_0, e_1, \ldots, e_{n-1})$. The following are well known to hold in any Post algebra:

(I) The set $E = \{e_0, e_1, \ldots, e_{n-1}\}$ is a chain $0 = e_0 < e_1 < \ldots < e_{n-1} = 1$ having the property that e_i is the smallest dense element in $[e_{i-1}, 1]$.

(II) $\langle H;\ \wedge,\ \vee,\ *;\ 0\rangle$ is a Heyting algebra.

(III) $D_i(x \wedge y) = D_i(x) \wedge D_i(y)$, $D_i(x \vee y) = D_i(x) \vee D_i(y)$.

(IV) $D_i(x) = (e_i * x)^{++}$.

(V) $D_i(e_j) = 1$ for $i \leqslant j$ and $D_i(e_j) = 0$ for $i > j$ $(0 \leqslant j \leqslant n-1)$.

(VI) $e_{i-1} \vee D_i(x) = (x * e_{i-1}) * e_{i-1}$.

3. Structure of $\mathfrak{M}(H)$. If H is a Heyting algebra and $f, g \in \mathfrak{M}(H)$. then $\mathfrak{M}(H)$ becomes a meet-semilattice, $f \wedge g$ being defined pointwise, Moreover, $\mathfrak{M}(H)$ is bounded with $0 \in \mathfrak{M}(H)$ defined by $0(x) = x$ and with $1 \in \mathfrak{M}(H)$ defined by $1(x) = 1$. In general, $\mathfrak{M}(H)$ is neither a lattice nor a pseudocomplemented semilattice. If, however, H is complete and $\{f_a;\ a \in A\} \subseteq \mathfrak{M}(H)$, then $\bigwedge_a f_a$ defined pointwise is modal, and so $\mathfrak{M}(H)$ is a complete lattice. The join in $\mathfrak{M}(H)$ will be denoted by $\cup$. If $f, g \in \mathfrak{M}(H)$, then the function $f * g$ defined pointwise is not, in general, modal. Nevertheless, we show that if H is a complete Heyting algebra, then $\mathfrak{M}(H)$ is a Heyting algebra. We start with some simple observations.

LEMMA 1. *If H is a Heyting algebra and $a \in H$, then the operators u_a, v_a, w_a, defined by*

$$u_a(x) = a \vee x, \qquad v_a(x) = a * x, \qquad w_a(x) = (x * a) * a,$$

are all modal.

LEMMA 2. *If H is a Heyting algebra and $f \in \mathfrak{M}(H)$, then*

$$f(x*y) \leqslant f(x)*f(y) = x*f(y) \quad \text{for all } x, y \in H.$$

Proof. Since $x \wedge (x*y) \leqslant y$ and f preserves meets, we have

$$f(x) \wedge f(x*y) \leqslant f(y),$$

and so

$$f(x*y) \leqslant f(x)*f(y).$$

Next observe that

$$f(x)*f(y) \leqslant x*f(y),$$

since f is inflationary. Moreover,

$$x*f(y) \leqslant f\big(x*f(y)\big) \leqslant f(x)*f\big(f(y)\big) = f(x)*f(y),$$

and so

$$f(x)*f(y) = x*f(y).$$

LEMMA 3. *If H is a Heyting algebra, $f \in \mathfrak{M}(H)$ and H_f is the image of H under f, then $\bigwedge \{w_a;\, a \in H_f\}$ exists and is f.*

Proof. If $x \in H$, then, since f is inflationary, $f(x) = w_a(x)$, where $a = f(x) \in H_f$. Moreover, for any $a \in H_f$ we have $a = f(a)$ so that

$$f(x) \leqslant \big(f(x)*f(a)\big)*f(a) = \big(x*f(a)\big)*f(a) = (x*a)*a = w_a(x)$$

by Lemma 2. Therefore, $f(x)$ is the least element in $\{w_a(x);\, a \in H_f\}$ and $\bigwedge \{w_a;\, a \in H_f\}$ exists and is f.

LEMMA 4. *If H is a complete Heyting algebra and $f: H \to H$ is inflationary and idempotent, then $f^+ = \bigwedge \{w_a;\, a \in H_f\}$ is the largest modal operator below f.*

Proof. For any $x \in H$ we have $a = f(x) \in H_f$ and $f^+\big(f(x)\big) \leqslant w_a\big(f(x)\big) = f(x)$, so that $f^+f \leqslant f$. Since f^+ is inflationary, $f \leqslant f^+f$ and, therefore, $f^+f = f$. However, f is also inflationary, and so $f^+ \leqslant f^+f = f$. Thus, $f^+ \in \mathfrak{M}(H)$ and $f^+ \leqslant f$. Now suppose that $g \in \mathfrak{M}(H)$, $g \leqslant f$ and $a \in H_f$; then $g(a) \leqslant f(a) = a$ so that $a = g(a)$ and, therefore, $a \in H_g$. It follows from Lemma 3 that $g \leqslant f^+$, and so f^+ is the largest modal operator below f.

LEMMA 5. *If H is a Heyting algebra and $f, g \in \mathfrak{M}(H)$, then $f*g$ defined pointwise is inflationary and idempotent.*

Proof. We have

$$(f*g)(x) = f(x)*g(x) \geqslant g(x) \geqslant x$$

so that $f*g$ is inflationary. For idempotency we use Lemma 2 to assert that

$$\begin{aligned}(f*g)(f*g)(x) &= f\big(f(x)*g(x)\big)*g\big(f(x)*g(x)\big) \leqslant f\big(f(x)*g(x)\big)*\big(g\,(f(x))*g(x)\big)\\ &\leqslant \big\{f\big(f(x)*g(x)\big) \wedge f(x)\big\}*g(x) = f(x)*g(x) = (f*g)(x).\end{aligned}$$

If we denote the largest modal operator below $f*g$ by $f \boxtimes g$, then we have the following

THEOREM 1. *If H is a complete Heyting algebra, then so is*

$$\langle \mathfrak{M}(H);\ \wedge, \cup, \boxtimes, 0, 1\rangle.$$

Proof. Observe that

$$f\wedge(f\boxtimes g)\leqslant f\wedge(f*g)=f\wedge g\leqslant g.$$

Moreover, if $h\in\mathfrak{M}(H)$ and $f\wedge h\leqslant g$, then $h\leqslant f*g$, and so $h\leqslant f\boxtimes g$.

Our next objective is the derivation of necessary and sufficient conditions for the modal extension of a complete Heyting algebra to be a Boolean algebra. In this connection, we need some preliminary results concerning function composition of modal operators. Generally speaking, the usual function composition fg of two modal operators f and g need not be modal; contrariwise, it is easy to see that $f\cup g$ exists and is fg.

LEMMA 6. *If H is a Heyting algebra, $a\in H$ and $f\in\mathfrak{M}(H)$, then $fu_a\in\mathfrak{M}(H)$ and $fu_a=f\cup u_a$.*

Proof. Clearly, fu_a is inflationary. Idempotency follows from the observation that

$$(fu_a)(fu_a)(x)=f\big(a\vee f(a\vee x)\big)=f\big(f(a\vee x)\big)=fu_a(x).$$

Finally,

$$\begin{aligned} fu_a(x\wedge y) &= f\big(a\vee(x\wedge y)\big)=f\big((a\vee x)\wedge(a\vee y)\big)\\ &= f(a\vee x)\wedge f(a\vee y)=fu_a(x)\wedge fu_a(y)\end{aligned}$$

so that fu_a is meet-preserving.

LEMMA 7. *If H is a Heyting algebra and $f\in\mathfrak{M}(H)$, then $w_0f=w_0u_{f(0)}$.*

Proof. Since

$$f(x)\wedge x^*\leqslant f\big(f(x)\wedge x^*\big)=f(x)\wedge f(x^*)=f(x\wedge x^*)=f(0),$$

we have $f(x)\wedge x^*\wedge f(0)^*=0$ so that

$$f(x)\leqslant\big(x^*\wedge f(0)^*\big)^*=\big(x\vee f(0)\big)^{**}$$

and, therefore,

$$f(x)^{**}\leqslant\big(x\vee f(0)\big)^{**}.$$

Moreover, $x\vee f(0)\leqslant f(x)$ so that $\big(x\vee f(0)\big)^{**}\leqslant f(x)^{**}$ and the result follows.

COROLLARY 1. *If H is a Boolean algebra, then every modal operator on H is of the form u_a and $H\cong\mathfrak{M}(H)$.*

LEMMA 8. *If H is a Heyting algebra and $a\in H$, then u_a and v_a are complemented in $\mathfrak{M}(H)$ and each is the complement of the other.*

Proof. We have

$$(u_a\wedge v_a)(x)=(a\vee x)\wedge(a*x)=\{a\wedge(a*x)\}\vee\{x\wedge(a*x)\}=(a\wedge x)\vee x=x.$$

Also, by Lemma 6, $v_au_a\in\mathfrak{M}(H)$ so that $u_a\cup v_a$ exists and

$$(u_a\cup v_a)(x)=(v_au_a)(x)=a*(a\vee x)=1.$$

LEMMA 9. *Let H be a Heyting algebra. Then w_0 is complemented in $\mathfrak{M}(H)$ if and only if H has the smallest dense element. If H has the smallest dense element e, then $w_0 = v_e$.*

Proof. First observe that if $f \in \mathfrak{M}(H)$, then, by Lemmas 6 and 7, $w_0 f \in \mathfrak{M}(H)$ so that $w_0 \cup f$ exists and is $w_0 u_{f(0)}$. Therefore, if w_0 is complemented in $\mathfrak{M}(H)$, then there exists $f \in \mathfrak{M}(H)$ such that $w_0 u_{f(0)} = 1$ and $w_0 \wedge f = 0$. Now, $w_0 u_{f(0)} = 1$ if and only if $f(0) \in D(H)$, and $w_0 \wedge f = 0$ if and only if $x^{**} \wedge f(x) = x$ identically. It follows that H_f contains $D(H)$ and, therefore, $f(0)$ is the smallest dense element in H. Finally, note that if e is the smallest dense element in H, then

$$\begin{aligned} v_e(x) &= e * x = e * \big(x^{**} \wedge (x \vee x^*)\big) = (e * x^{**}) \wedge \big(e * (x \vee x^*)\big) = e * x^{**} \\ &= (e \wedge x^*)^* = (e \wedge x^*)^{***} = (e^{**} \wedge x^*)^* = x^{**} = w_0(x). \end{aligned}$$

LEMMA 10. *Let H be a Heyting algebra and $a \in H$. If w_a is complemented in $\mathfrak{M}(H)$, then $[a, 1]$ has the smallest dense element. If H is complete and $[a, 1]$ has a smallest dense element, then w_a is complemented in the complete Heyting algebra $\mathfrak{M}(H)$.*

Proof. For each $f \in \mathfrak{M}([a, 1])$, define $f^+ : H \to H$ by $f^+ = f u_a$. Observe that $f^+ \in \mathfrak{M}(H)$, $f^+ | [a, 1] = f$ and if $f, g \in \mathfrak{M}([a, 1])$ are such that $f \leqslant g$, then $f^+ \leqslant g^+$. Note also that if $f \in \mathfrak{M}(H)$, then

$$f | [a, 1] \in \mathfrak{M}([a, 1]) \quad \text{and} \quad (f | [a, 1])^+ \geqslant f.$$

Now, if w_a is complemented in $\mathfrak{M}(H)$, then there exists $f \in \mathfrak{M}(H)$ such that $w_a \wedge f = 0$ and $w_a \cup f$ exists and is 1. But if

$$g \in \mathfrak{M}([a, 1]) \quad \text{and} \quad w_a | [a, 1], f | [a, 1] \leqslant g,$$

then

$$w_a \leqslant (w_a | [a, 1])^+ \leqslant g^+ \quad \text{and} \quad f \leqslant (f | [a, 1])^+ \leqslant g^+.$$

Therefore, we have $g^+ = 1$ so that $g = g^+ | [a, 1] = 1$. Thus the join $w_a | [a, 1] \cup_a f | [a, 1]$ of $w_a | [a, 1]$ and $f | [a, 1]$ exists in $\mathfrak{M}([a, 1])$ and is 1. We also have

$$w_a | [a, 1] \wedge f | [a, 1] = 0,$$

and so $w_a | [a, 1]$ is complemented in $\mathfrak{M}([a, 1])$. In other words, the double pseudocomplementation operator on the Heyting algebra $[a, 1]$ is complemented in $\mathfrak{M}([a, 1])$ and so, by Lemma 9, $[a, 1]$ has a smallest dense element.

If H is complete and $[a, 1]$ has the smallest dense element e_a, then $w_a | [a, 1] = \bar{v}_{e_a}$, where $\bar{v}_{e_a} \in \mathfrak{M}([a, 1])$ is defined by $\bar{v}_{e_a}(x) = e_a * x$ for all $x \in [a, 1]$, that is

$$(x * a) * a = e_a * x \quad \text{for all } x \in [a, 1].$$

Consequently, $\big((x \vee a) * a\big) * a = e_a * (x \vee a)$ for all $x \in H$. However,

$$\big((x \vee a) * a\big) * a = \big((x * a) \wedge (a * a)\big) * a = (x * a) * a$$

holds in any Heyting algebra H. Thus,

$$w_a = v_{e_a} u_a = v_{e_a} \cup u_a$$

which, since v_{e_a} and u_a are complemented in the complete Heyting algebra $\mathfrak{M}(H)$, shows that w_a is complemented in $\mathfrak{M}(H)$.

THEOREM 2. *Let H be a complete Heyting algebra. Then $\mathfrak{M}(H)$ is a Boolean algebra if and only if $[a, 1]$ has a smallest dense element for all $a \in H$.*

Proof. If every interval $[a, 1]$ in a complete Heyting algebra H has a smallest dense element and $f \in \mathfrak{M}(H)$, then $f \in B(\mathfrak{M}(H))$, since $f = \bigwedge \{w_a;\ a \in H_f\}$, $w_a \in B(\mathfrak{M}(H))$ and $B(\mathfrak{M}(H))$ is closed under arbitrary meets in $\mathfrak{M}(H)$. Thus, $\mathfrak{M}(H) = B(\mathfrak{M}(H))$ which is a Boolean algebra.

COROLLARY 1. *If H is a Heyting algebra satisfying the descending chain condition, then $\mathfrak{M}(H)$ is a Boolean algebra.*

The corollary follows immediately from the fact that any distributive lattice satisfying the descending chain condition is a complete lattice in which every filter is principal.

COROLLARY 2. *If H is a complete Post algebra, then $\mathfrak{M}(H)$ is a Boolean algebra.*

Proof. Straightforward calculations show that in any Post algebra of order $n \geqslant 2$ the element

$$e_a = e_1 \vee \bigvee_{i=2}^{n-1} D_{i-1}(a) \wedge e_i$$

is the smallest dense element in the interval $[a, 1]$.

Remark. Corollary 2 will be generalized to arbitrary Post algebras in the next section.

4. Modal extensions of Post algebras. Prior to obtaining a normal form for modal operators on a Post algebra, recall that an *algebraic function of m variables* on a Post algebra H of order $n \geqslant 2$ is one obtained from the constant functions $a(x_1, x_2, \ldots, x_m) = a$ and projection functions $p_i(x_1, x_2, \ldots, x_m) = x_i$ by a finite number of applications of the operations $\wedge, \vee, *, D_1, D_2, \ldots, D_{n-1}$. In [2], it is shown that a function $f: H^m \to H$ is algebraic if and only if it has the *congruence substitution property*:

$a_i \equiv b_i(\theta)$ $(1 \leqslant i \leqslant m)$ implies $f(a_1, a_2, \ldots, a_m) \equiv f(b_1, b_2, \ldots, b_m)(\theta)$ for all congruences θ on H.

Furthermore, if $f, g: H^m \to H$ are algebraic, then $f = g$ identically if and only if f and g agree on the chain of constants E. With these remarks in mind we prove the following

LEMMA 11. *If H is a Post algebra of order $n \geqslant 2$ and $f \in \mathfrak{M}(H)$, then f is algebraic.*

Proof. It is well known (see [2]) that there are an isomorphism of the lattice of congruences θ on H and a lattice of Post filters F (lattice filters closed under D_{n-1}) under the correspondence

$$F = \{x \in H;\, x \equiv 1(\theta)\},$$
$$\theta = \{(x, y) \in H^2;\, x \wedge a = y \wedge a \text{ for some } a \in F\}.$$

Thus, if $x \equiv y(\theta)$, then $x \wedge a = y \wedge a$ for some $a \in F$, so that $f(x) \wedge f(a) = f(y) \wedge f(a)$, which shows, since $f(a) \geqslant a \in F$, that $f(x) \equiv f(y)(\theta)$. Hence, any $f \in \mathfrak{M}(H)$ has the congruence substitution property and is therefore algebraic.

THEOREM 3. *Let H be a Post algebra of order $n \geqslant 2$. Then $f : H \to H$ is a modal operator if and only if it can be (uniquely) represented in the form*

$$f(x) = \bigvee_{i=0}^{n-1} a_i \wedge D_i(x),$$

where the relations $a_i \geqslant a_{i-1} \vee e_i$ $(1 \leqslant i \leqslant n-1)$, $D_i(a_k) \leqslant a_i * a_k$ hold for all $i, k \in \{0, 1, \ldots, n-1\}$ (and D_0 denotes the operator 1).

Proof. If $f \in \mathfrak{M}(H)$, then, by Lemma 11, f is algebraic. Furthermore, since f is monotone, it follows from [2] that

$$f(x) = \bigvee_{i=0}^{n-1} f(e_i) \wedge D_i(x)$$

identically and is therefore of the form $\bigvee_{i=0}^{n-1} a_i \wedge D_i(x)$, where $a_i \geqslant a_{i-1} \vee e_i$ $(1 \leqslant i \leqslant n-1)$, f being inflationary and monotone. Now observe that $f(f(x)) \leqslant f(x)$ if and only if $f(f(e_k)) \leqslant f(e_k)$ for all $k \in \{0, 1, \ldots, n-1\}$ which, in turn, is equivalent to

$$f(a_k) = \bigvee_{i=0}^{n-1} a_i \wedge D_i(a_k) \leqslant a_k \quad \text{for all } k \in \{0, 1, \ldots, n-1\}.$$

Therefore, $f(f(x)) \leqslant f(x)$ holds if and only if $D_i(a_k) \leqslant a_i * a_k$ for all $i, k \in \{0, 1, \ldots, n-1\}$.

If, conversely,

$$f(x) = \bigvee_{i=0}^{n-1} a_i \wedge D_i(x)$$

identically, where $a_i \geqslant a_{i-1} \vee e_i$ $(1 \leqslant i \leqslant n-1)$ and $D_i(a_k) \leqslant a_i * a_k$ for all $i, k \in \{0, 1, \ldots, n-1\}$, then

$$f(e_k) = \bigvee_{i=0}^{n-1} a_i \wedge D_i(e_k) = \bigvee_{i \leqslant k} (a_i \wedge D_i(e_k)) \vee \bigvee_{i \leqslant k} (a_i \wedge D_i(e_k)) = \bigvee_{i \leqslant k} a_i = a_k.$$

Consequently, the operator f is inflationary, since $f(e_i) = a_i \geqslant e_i$ for all $i \in \{0, 1, \ldots, n-1\}$. To show that f is idempotent, it suffices, since f is inflationary, to show that $f(f(e_k)) \leqslant f(e_k)$ for all $k \in \{0, 1, \ldots, n-1\}$. However, since $f(e_k) = a_k$, it follows from the above that this holds if and only if $D_i(a_k) \leqslant a_i * a_k$ for all $i, k \in \{0, 1, \ldots, n-1\}$. Finally, f is meet-preserving if and only if

$$f(e_i \wedge e_k) = f(e_i) \wedge f(e_k) \quad \text{for all } i, k \in \{0, 1, \ldots, n-1\}$$

which holds since $e_0 \leqslant e_1 \leqslant \ldots \leqslant e_{n-1}$ and f is obviously monotone. The uniqueness of the representation is clear from the proof.

THEOREM 4. *If H is a Post algebra of order $n \geqslant 2$, then $\mathfrak{M}(H)$ is a Boolean algebra.*

Proof. We start by showing that $\mathfrak{M}(H)$ is a pseudocomplemented semilattice. Let $f, g \in \mathfrak{M}(H)$ have normal forms

$$f(x) = \bigvee_{i=0}^{n-1} a_i \wedge D_i(x) \quad \text{and} \quad g(x) = \bigvee_{i=0}^{n-1} b_i \wedge D_i(x).$$

Now, $f \wedge g = 0$ if and only if $f(x) \wedge g(x) = x$ identically or, equivalently, $f(e_i) \wedge g(e_i) \leqslant e_i$ for $i \in \{0, 1, \ldots, n-1\}$. It follows from the equality $f(e_i) = a_i$ that

$f \wedge g = 0$ if and only if $b_i \leqslant a_i * e_i$ for all $i \in \{0, 1, \ldots, n-1\}$,

or, equivalently,

$$b_i \leqslant \bigwedge_{k=i}^{n-1} (a_k * e_k) \quad \text{for all } i \in \{0, 1, \ldots, n-1\},$$

since $b_0 \leqslant b_1 \leqslant \ldots \leqslant b_{n-1}$. Define $f^* : H \to H$ by

$$f^*(x) = \bigvee_{i=0}^{n-1} \Big(\bigwedge_{k=i}^{n-1} (a_k * e_k) \Big) \wedge D_i(x).$$

Clearly, if $f^* \in \mathfrak{M}(H)$, then it is the pseudocomplement of f in $\mathfrak{M}(H)$. To see that $f^* \in \mathfrak{M}(H)$, first note that if

$$c_i = \bigwedge_{k=i}^{n-1} (a_k * e_k),$$

then

$$c_i \geqslant \bigwedge_{k=i}^{n-1} e_k = e_i \quad \text{and} \quad c_i \geqslant c_{i-1} \quad \text{for } i \in \{1, 2, \ldots, n-1\}.$$

It remains to show that $D_j(c_k) \leqslant c_j * c_k$ for all $j, k \in \{0, 1, \ldots, n-1\}$. Clearly, $c_j \leqslant c_k$ whenever $j \leqslant k$ so that $D_j(c_k) \leqslant c_j * c_k$ whenever $j \leqslant k$. Suppose that $j > k$. Then

$$\begin{aligned} D_j(c_k) &= D_j\Big(\bigwedge_{i=k}^{n-1} (a_i * e_i)\Big) = \bigwedge_{i=k}^{n-1} D_j(a_i * e_i) = \bigwedge_{i=k}^{n-1} \{e_j * (a_i * e_i)\}^{++} \\ &= \bigwedge_{i=k}^{n-1} \{a_i * (e_j * e_i)\}^{++} = \bigwedge_{i=k}^{j-1} \{a_i * (e_j * e_i)\}^{++} \wedge \bigwedge_{i=j}^{n-1} \{a_i * (e_j * e_i)\}^{++} \\ &= \bigwedge_{i=k}^{j-1} (a_i * e_i)^{++}, \end{aligned}$$

since $e_j * e_i = e_i$ if $j > i$ and $e_j * e_i = 1$ if $j \leqslant i$. Also,

$$\begin{aligned} c_j * c_k &= c_j * \bigwedge_{m=k}^{n-1} (a_m * e_m) = c_j * \Big\{ \bigwedge_{m=k}^{j-1} (a_m * e_m) \wedge c_j \Big\} = c_j * \bigwedge_{m=k}^{j-1} (a_m * e_m) \\ &\geqslant \bigwedge_{m=k}^{j-1} (a_m * e_m) \geqslant \bigwedge_{m=k}^{j-1} (a_m * e_m)^{++} = D_j(c_k). \end{aligned}$$

Therefore, $f^* \in \mathfrak{M}(H)$.

Thus, in proving the theorem, it suffices to show that $f = f^{**}$ for any $f \in \mathfrak{M}(H)$. By the uniqueness of the normal form for modal operators we need only to show that

$$a_i = \bigwedge_{k=i}^{n-1} \Big\{\Big(\bigwedge_{m=k}^{n-1} (a_m * e_m)\Big) * e_k\Big\} \quad \text{for all } i \in \{0, 1, \ldots, n-1\}.$$

Using the identity $(x \wedge y) * z = (x * z) \vee (y * z)$ which is valid in any Post algebra (in fact, in any L-algebra) we see that

$$\Big(\bigwedge_{m=k}^{n-1} (a_m * e_m)\Big) * e_k = \bigvee_{m=k}^{n-1} \{(a_m * e_m) * e_k\} = \{(a_k * e_k) * e_k\} \vee \bigvee_{m=k+1}^{n-1} \{(a_m * e_m) * e_k\}.$$

However, since $a_m * e_m \geqslant e_{k+1}$ (whenever $m \geqslant k+1$) and e_{k+1} is the smallest dense element in $[e_k, 1]$, the element $a_m * e_m$ is dense in $[e_k, 1]$, and so $(a_m * e_m) * e_k = e_k$ whenever $m \geqslant k+1$. Therefore,

$$\Big(\bigwedge_{m=k}^{n-1} (a_m * e_m)\Big) * e_k = (a_k * e_k) * e_k,$$

and so $f = f^{**}$ if and only if

$$a_i = \bigwedge_{k=i}^{n-1} w_{e_k}(a_k) \quad \text{for all } i \in \{0, 1, \ldots, n-1\}.$$

The last condition is easily seen to be equivalent to the a_i being given by

$$a_{n-1} = 1 \quad \text{and} \quad a_{i-1} = w_{e_{i-1}}(a_{i-1}) \wedge a_i \quad \text{for } i \in \{1, 2, \ldots, n-1\}.$$

Next observe that, since $a_{i-1} \leqslant w_{e_{i-1}}(a_{i-1}) \wedge a_i$,

$$a_{i-1} = w_{e_{i-1}}(a_{i-1}) \wedge a_i \text{ if and only if } w_{e_{i-1}}(a_{i-1}) \leqslant a_i * a_{i-1}.$$

Since

$$w_{e_{i-1}}(a_{i-1}) = e_{i-1} \vee D_i(a_{i-1}),$$

$$e_{i-1} \leqslant a_i * a_{i-1} \quad \text{and} \quad D_i(a_k) \leqslant a_i * a_k,$$

we have $f = f^{**}$.

In order to extend the last result to the class of P-algebras of type $n \geqslant 2$, we need the following

LEMMA 12. *If H_1 and H_2 are Heyting algebras, then*

$$\mathfrak{M}(H_1 \times H_2) \cong \mathfrak{M}(H_1) \times \mathfrak{M}(H_2).$$

Proof. Let $f_i \in \mathfrak{M}(H_i)$ $(i = 1, 2)$. Define

$$\varphi : \mathfrak{M}(H_1) \times \mathfrak{M}(H_2) \to \mathfrak{M}(H_1 \times H_2)$$

by

$$\varphi(f_1, f_2) = f_1 \times f_2,$$

where $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for all $(x_1, x_2) \in H_1 \times H_2$. Clearly, φ is a well-defined order embedding, and so it remains to show that φ is onto $\mathfrak{M}(H_1 \times H_2)$. Let $p_i : H_1 \times H_2 \to H_i$ $(i = 1, 2)$ be the projection homomorphisms. Given $f \in \mathfrak{M}(H_1 \times H_2)$, define $f_i : H_i \to H_i$ by

$$f_i(x_i) = p_i f(x_1, x_2) \qquad (i = 1, 2).$$

We show that f_1 and f_2 are well defined, belong to $\mathfrak{M}(H_1)$ and $\mathfrak{M}(H_2)$, respectively, and that $f = f_1 \times f_2$. First observe that

$$f_1(x_1) = p_1 f(x_1, x_2) = p_1\big(f(x_1, 1) \wedge f(1, x_2)\big) = p_1 f(x_1, 1) \wedge p_1 f(1, x_2)$$

and, since f is inflationary and p_1 is monotone, $p_1 f(1, x_2) \geqslant p_1(1, x_2) = 1$ so that $f_1(x_1) = p_1 f(x_1, 1)$. It follows that f_1 is well defined. Similarly we can show that $f_2(x_2) = p_2 f(1, x_2)$ and so f_2 is well defined. To see that $f_1 \in \mathfrak{M}(H_1)$, observe that

$$x_1 = p_1(x_1, 1) \leqslant p_1 f(x_1, 1) = f_1(x_1),$$

and so f_1 is inflationary. Also f_1 preserves meets, since so do p_1 and f. To prove that f_1 is idempotent, it suffices to show that $f_1\big(f_1(x_1)\big) \leqslant f_1(x_1)$. To see this, observe that

$$f_1\big(f_1(x_1)\big) * f_1(x) = p_1 f\big(f_1(x_1), 1\big) * p_1 f(x_1, 1) = p_1\big(f\big(f_1(x_1), 1\big) * f(x_1, 1)\big)$$
$$= p_1\big(\big(f_1(x_1), 1\big) * f(x_1, 1)\big) = p_1\big(f_1(x_1), 1\big) * p_1 f(x_1, 1) = f_1(x_1) * f_1(x_1) = 1$$

and the result follows. Similarly we can show that $f_2 \in \mathfrak{M}(H_2)$. Finally, $f = f_1 \times f_2$, for

$$(f_1 \times f_2)(x_1, x_2) = \big(f_1(x_1), f_2(x_2)\big) = \big(p_1 f(x_1, 1),\ p_2 f(1, x_2)\big)$$
$$= \big(p_1 f(x_1, x_2), p_2 f(x_1, x_2)\big) = f(x_1, x_2).]$$

LEMMA 13. *Let H be a Heyting algebra. Then H is an L-algebra of type $n \geqslant 2$ if and only if the centre of H is a subalgebra of H and $D(H)$ is a K-algebra of type $n-1$.*

For the proof see [9].

THEOREM 5. *If H is a P-algebra of type $n \geqslant 2$, then the following are equivalent:*

(1) *$\mathfrak{M}(H)$ is a Boolean algebra.*

(2) *$[a, 1]$ has a smallest dense element for all $a \in H$.*

(3) *There exists a chain $0 = e_0 \leqslant e_1 \leqslant \ldots \leqslant e_{n-1} = 1$ in L such that e_i is the smallest dense element in $[e_{i-1}, 1]$ $(1 \leqslant i \leqslant n-1)$.*

(4) *H is a P_2-algebra of order at most n.*

(5) *H is a product of finitely many Post algebras each of order at most n.*

Proof. Let H be a P-algebra of type $n \geqslant 2$. By Lemma 10, (1) implies (2). Furthermore, (2) implies that there exists a chain $0 = e_0 \leqslant e_1 \leqslant \ldots$ in H such that e_i is the smallest dense element in $[e_{i-1}, 1]$. Since H is an L-algebra of type $n \geqslant 2$, the repeated application of Lemma 13 shows that $[e_{n-2}, 1]$ is an L-algebra of type 2. Equivalently, $[e_{n-2}, 1]$ is a Boolean algebra, and so $e_{n-1} = 1$. Consequently, H is a Stone lattice of order at most n whose dual is a Stone algebra. Therefore, H is a P_2-algebra of order at most n. It follows from [4] that H is a product of finitely many Post algebras of order at most n. Finally, if (5) holds, then, by Theorem 5 and Lemma 12, $\mathfrak{M}(H)$ is a Boolean algebra.

COROLLARY 1. *Let H be a three-valued Łukasiewicz algebra. Then $\mathfrak{M}(H)$ is a Boolean algebra if and only if H has a smallest dense element.*

Proof. It is known that any three-valued Łukasiewicz algebra H is a *, $^+$-subalgebra of a product of two- and three-element chains. Furthermore, Varlet [11] has shown that H is a Heyting algebra and a dual Heyting algebra in which $a*b = (a^*\vee b^{**})\wedge(a^+\vee b)$, and a similar formula holds for the dual operation. Consequently, any three-valued Łukasiewicz algebra is a P-algebra of type 3. It remains only to show that if e_1 is the smallest dense element in H, then 1 is the smallest dense element in $[e_1, 1]$. To see this, suppose that $x \geqslant e_1$ and $x*e_1 = e_1$. Since $x*e_1 = x^+\vee e_1$, we have $x^+ \leqslant e_1 \leqslant x$ and, therefore, $x = 1$.

REFERENCES

[1] R. Balbes and Ph. Dwinger, *Distributive lattices*, University of Missouri Press 1974.

[2] R. Beazer, *Some remarks on Post algebras*, Colloquium Mathematicum 29 (1974), p. 167-178.

[3] — *Post-like algebras and injective Stone algebras*, Algebra Universalis 5 (1975), p. 16-23.

[4] G. Epstein and A. Horn, *Chain-based lattices*, Pacific Journal of Mathematics 55 (1974), p. 65-84.

[5] — *P-algebras, an abstraction from Post algebras*, Algebra Universalis 4 (1974), p. 195-206.

[6] G. Grätzer, *Lattice theory: First concepts and distributive lattices*, San Francisco 1971.

[7] T. Hecht and T. Katriňák, *Equational classes of relative Stone algebras*, Notre Dame Journal of Formal Logic 13 (1972), p. 248-254.

[8] A. Horn, *Logic with truth values in a linearly ordered Heyting algebra*, Journal of Symbolic Logic 34 (1969), p. 395-408.

[9] T. Katriňák and A. Mitschke, *Stonesche Verbände der Ordnung n und Post Algebren*, Mathematische Annalen 199 (1972), p. 13-30.

[10] D. S. Macnab, *An algebraic study of modal operators on Heyting algebras with applications to topology and sheafification*, Ph. D. thesis, University of Aberdeen 1976.

[11] J. Varlet, *Considérations sur les algèbres de Łukasiewicz trivalentes*, Bulletin de la Société Royale des Sciences de Liège 38 (1969), p. 462-469.

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