COLLOQUIUM MATHEMATICUM

VOL. XXXV 1976 FASC. 1

SOME COMMENTS ON INDEPENDENT σ-ALGEBRAS

BY

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Let X be a non-empty set and suppose $\mathscr{A}$ and $\mathscr{B}$ are σ-algebras of subsets of X. Given probability measures μ and ν on $\langle X, \mathscr{A}\rangle$ and $\langle X, \mathscr{B}\rangle$, respectively, we will say that a measure η on $\langle X, \mathscr{C}\rangle$, $\mathscr{C} = \sigma(\mathscr{A}\cup\mathscr{B})$, is a *splicing* of μ and ν if

$$(1) \qquad \eta(A\cap B) = \mu(A)\nu(B) \quad \text{for all } A\in\mathscr{A} \text{ and } B\in\mathscr{B}.$$

It is easily seen that there is at most one such splicing. Hence, the question is one of existence.

This question has been studied by several authors (cf., for instance, [1]-[3], [5], and [6]). In particular, it was asked by Marczewski [4] whether the condition

$$(2) \qquad A\cap B = \emptyset \Rightarrow \mu(A)\nu(B) = 0$$

is not necessary and sufficient for the existence of a splicing. The necessity of (2) is obvious. However, Helson [2] provided an example which shows that it is not sufficient. Since then, Marczewski [3] has shown that (2) is sufficient if $\mathscr{A}$ is a finite σ-algebra.

The purpose of the present paper* is to investigate this problem a little further. First we will prove (cf. the Theorem below) that a necessary and sufficient condition for a splicing to exist is that

$$(3) \qquad X = \bigcup_1^\infty A_n \cap B_n \Rightarrow \sum_1^\infty \mu(A_n)\nu(B_n) \geqslant 1.$$

Condition (3) is equivalent to the condition given by Sikorski [6] in his Theorem 8. We have included here a proof of (3) which is independent of Sikorski's result on the grounds that our proof is extremely short and simple and has the advantage that it does not require any preparation. In particular, our proof does not use the fact (proved by Marczew-

* This work was partially supported by N.S.F. grant GP-4383.

ski [3]) that a finitely additive splicing always exists under condition (2), a result which is an easy consequence of the Theorem (cf. Corollary 2).

Next, we use the Theorem to prove various results related to Marczewski's original questions. One such result is (cf. Corollary 3) that a splicing exists if

$$(4)\qquad A\cap B=\emptyset \Rightarrow A=\emptyset \quad\text{or}\quad \nu(B)=0.$$

This condition was announced in [4] by Marczewski. Another application of the Theorem is to the notion of conditionally independent σ-algebras. We will say $\mathscr{A}$ and $\mathscr{B}$ are *conditionally independent*, given the σ-algebra $\mathscr{D}\subseteq \mathscr{A}\cap\mathscr{B}$, if

$$(5)\qquad A\cap B=\emptyset \Rightarrow (\forall x\in A)(\exists D\in\mathscr{D})\; x\in D \text{ and } D\cap B=\emptyset.$$

Under the assumption that $\mathscr{D}$ is countably generated, we will show (cf. Corollary 4) that μ and ν admit a splicing if and only if

$$(6)\qquad (\forall D\in\mathscr{D})\mu(D)=\nu(D)\in\{0,1\}.$$

In this situation, it will be shown that (6) is equivalent to (2). Finally, we will provide an example which shows that (6) is not sufficient, in general, when $\mathscr{D}$ is not countably generated. It turns out that the example constructed for this purpose provides another proof that (2) is not sufficient.

We are grateful to the referee for bringing to our attention some of the literature on this subject. In particular, it was he who pointed out the relation of our work to Sikorski's.

THEOREM. *Condition* (3) *is necessary and sufficient for a splicing of* μ *and* ν *to exist.*

Proof. Given $x\in X$, write

$$A(x)=\bigcap\{A\in\mathscr{A}:\ x\in A\}\quad\text{and}\quad B(x)=\bigcap\{B\in\mathscr{B}:\ x\in B\}.$$

Set $X_{\mathscr{A}}=\{A(x):\ x\in X\}$, $X_{\mathscr{B}}=\{B(x):\ x\in X\}$, and define $\Phi\colon X\to X_{\mathscr{A}}$ and $\Psi\colon X\to X_{\mathscr{B}}$ so that $\Phi(x)=A(x)$ and $\Psi(x)=B(x)$. It is easy to see that $\Phi^{-1}(\Phi(A))=A$ and $\Psi^{-1}(\Psi(B))=B$ for all $A\in\mathscr{A}$ and $B\in\mathscr{B}$. Hence, if

$$\tilde{\mathscr{A}}=\{\Phi(A):\ A\in\mathscr{A}\}\quad\text{and}\quad \tilde{\mathscr{B}}=\{\Psi(B):\ B\in\mathscr{B}\},$$

then Φ and Ψ determine σ-isomorphisms from $\langle X,\mathscr{A}\rangle$ and $\langle X,\mathscr{B}\rangle$ onto $\langle X_{\mathscr{A}},\tilde{\mathscr{A}}\rangle$ and $\langle X_{\mathscr{B}},\tilde{\mathscr{B}}\rangle$, respectively. Let $\tilde{\mu}=\mu^{\Phi^{-1}}$ and $\tilde{\nu}=\nu^{\Psi^{-1}}$, and consider the probability space $\langle X_{\mathscr{A}}\times X_{\mathscr{B}},\ \tilde{\mathscr{A}}\times\tilde{\mathscr{B}},\ \tilde{\mu}\times\tilde{\nu}\rangle$. Write

$$\tilde{X}=\{\langle A(x),B(x)\rangle:\ x\in X\},$$

$$\tilde{\mathscr{C}}=\{S\cap\tilde{X}:\ S\in\tilde{\mathscr{A}}\times\tilde{\mathscr{B}}\}\quad\text{and}\quad \Lambda=\Phi\times\Psi.$$

Then it is easy to see that Λ is a σ-isomorphism from $\langle X, \mathscr{C}\rangle$ onto $\langle \tilde{X}, \tilde{\mathscr{C}}\rangle$.

We will next show that there is a probability measure $\tilde{\eta}$ on $\langle \tilde{X}, \tilde{\mathscr{C}}\rangle$ such that $\tilde{\eta}(S\cap\tilde{X}) = \tilde{\mu}\times\tilde{\nu}(S)$ for $S\in\tilde{\mathscr{A}}\times\tilde{\mathscr{B}}$. In order to do this, it is sufficient to check that $\tilde{X}$ has $(\tilde{\mu}\times\tilde{\nu})$-outer measure 1. (In general, $\tilde{X}$ will not be $(\tilde{\mu}\times\tilde{\nu})$-measurable.) But if

$$\{\tilde{A}_n\}^\infty \subseteq \tilde{\mathscr{A}}, \quad \{\tilde{B}_n\}^\infty \subseteq \tilde{\mathscr{B}} \quad \text{and} \quad \tilde{X} \subseteq \bigcup_1^\infty \tilde{A}_n\times\tilde{B}_n,$$

then

$$X \subseteq \bigcup_1^\infty A_n \cap B_n, \quad \text{where } A_n = \Phi^{-1}(\tilde{A}_n) \text{ and } B_n = \Psi^{-1}(\tilde{B}_n).$$

Hence

$$\sum_1^\infty \tilde{\mu}(\tilde{A}_n)\tilde{\nu}(\tilde{B}_n) = \sum_1^\infty \mu(A_n)\nu(B_n) \geqslant 1,$$

and so $\tilde{X}$ has $(\tilde{\mu}\times\tilde{\nu})$-outer measure 1.

Finally, write $\eta(C) = \tilde{\eta}(\Lambda(C))$, $C\in\mathscr{C}$. Since Λ is a σ-isomorphism, η is a probability measure on $\langle X, \mathscr{C}\rangle$. Clearly,

$$\eta(A\cap B) = \tilde{\eta}(\Lambda(A\cap B)) = \tilde{\mu}(\Phi(A))\tilde{\nu}(\Psi(B)) = \mu(A)\nu(B).$$

COROLLARY 1. *Suppose $\mathscr{P}$ is a countable partitioning of X and that $\mathscr{A} = \sigma(\mathscr{P})$. Then* (2) *is necessary and sufficient for a splicing to exist.*

Proof. Let

$$X = \bigcup_1^\infty A_n\cap B_n$$

and put

$$A = \Big\{x\colon \sum_1^\infty \chi_{A_n}(x)\nu(B_n) < 1\Big\}.$$

We must show that $\mu(A) = 0$. Under the stated condition on $\mathscr{A}$, this will be proved once we show that every $x\in A$ is contained in a set of μ-measure 0. Given $x\in A$, set $N_x = \{n \geqslant 1\colon x\notin A_n\}$. Since

$$X = \bigcup_1^\infty A_n\cap B_n, \quad \Big(\bigcap_{n\in N_x} X\setminus A_n\Big)\cap\Big(\bigcap_{n\notin N_x} X\setminus B_n\Big) = \emptyset,$$

and so

$$\mu\Big(\bigcap_{n\in N_x} X\setminus A_n\Big) = 0,$$

since

$$\nu\Big(\bigcap_{n\notin N_x} X\setminus B_n\Big) = 1-\nu\Big(\bigcup_{n\notin N_x} B_n\Big) \geqslant 1-\sum_1^\infty \chi_{A_n}(x)\nu(B_n) > 0.$$

Since

$$x \in \bigcap_{n \in N_x} X \setminus A_n,$$

this completes the proof.

COROLLARY 2 (see also Marczewski [3]). *Condition* (2) *is necessary and sufficient for a finitely additive splicing to exist.*

COROLLARY 3. *Condition* (4) *is necessary and sufficient for* ν *to admit a splicing with every choice of* μ.

Proof. Again the necessity is easy. To prove the sufficiency, suppose

$$X = \bigcup_1^\infty A_n \cap B_n$$

and define A as in the proof of Corollary 1. We will show that $A = \emptyset$. Indeed, suppose $x \in A$ and write $N_x = \{n \geqslant 1 : x \notin A_n\}$. Then

$$\Big(\bigcap_{n \in N_x} X \setminus A_n\Big) \cap \Big(\bigcap_{n \notin N_x} X \setminus B_n\Big) = \emptyset \quad \text{and} \quad \nu\Big(\bigcap_{n \notin N_x} X \setminus B_n\Big) > 0.$$

But this means that

$$\bigcap_{n \in N_x} X \setminus A_n = \emptyset,$$

which is a contradiction.

COROLLARY 4. *Suppose* $\mathscr{A}$ *and* $\mathscr{B}$ *are conditionally independent, given the countably generated* σ*-algebra* $\mathscr{D} \subseteq \mathscr{A} \cap \mathscr{B}$. *Then the following are equivalent:*

(i) μ *and* ν *admit a splicing,*

(ii) $A \cap B = \emptyset \Rightarrow \mu(A)\nu(B) = 0$,

(iii) $(\forall D \in \mathscr{D})\ \mu(D) = \nu(D) \in \{0, 1\}$.

Proof. Clearly, (i) implies (ii). Moreover, it is easy to see that (ii) implies $\mu(S) = \nu(S) \in \{0, 1\}$ for all $S \in \mathscr{A} \cap \mathscr{B}$. Hence it only remains to show that (iii) implies (i).

Since $\mathscr{D}$ is countably generated, (iii) implies that there is an $x_0 \in X$ such that $\mu(D) = \nu(D) = \chi_D(x_0)$ for $D \in \mathscr{D}$. Let $D_0 = \bigcap\{D \in \mathscr{D} : x_0 \in \mathscr{D}\}$. Then

$$D_0 \in \mathscr{D}, \quad \mu(D_0) = \nu(D_0) = 1 \quad \text{and} \quad (\forall D \in \mathscr{D})\ D \cap D_0 \neq \emptyset \Rightarrow D_0 \subseteq D.$$

Now suppose that

$$X = \bigcup_1^\infty A_n \cap B_n.$$

We must show that

$$\sum_1^\infty \chi_{A_n}(x)\chi_{B_n}(y) \geqslant 1 \text{ a.e. with respect to } \mu \times \nu.$$

In particular, it is enough to prove it for $x, y \in D_0$. Given $x, y \in D_0$, suppose that

$$\sum_{1}^{\infty} \chi_{A_n}(x)\chi_{B_n}(y) = 0.$$

Write $N_1 = \{n \geqslant 1\colon x \notin A_n\}$ and $N_2 = \{n \geqslant 1\colon y \notin B_n\}$. Then we have $N_1 \cup N_2 = Z^+$, and so

$$\Big(\bigcap_{n\in N_1} X\setminus A_n\Big) \cap \Big(\bigcap_{n\in N_2} X\setminus B_n\Big) = \emptyset.$$

But

$$x \in A \equiv \bigcap_{n\in N_1} X\setminus A_n \quad \text{and} \quad y \in B = \bigcap_{n\in N_2} X\setminus B_n,$$

and if $x \in D \in \mathscr{D}$, then $y \in D$. Hence

$$(\forall D \in \mathscr{D})\, x \in D \Rightarrow B \cap D \neq \emptyset,$$

which is a contradiction.

Remark 1. The condition that $\mathscr{D}$ be countably generated is important, as the following example demonstrates. It should be noted that the importance of this condition does not lie in the measurability of atoms in $\mathscr{D}$, since atoms in $\mathscr{D}$ are $\mathscr{D}$-measurable in our example.

Let $\Omega = (\{0,1\})^{Z\setminus\{0\}}$. Given $\omega \in \Omega$, let $x(n,\omega)$ be the n-th coordinate of ω, $n \in Z\setminus\{0\}$. For $n \geqslant 1$, denote by $\mathscr{M}^{(n)}$ and $\mathscr{N}^{(n)}$ the σ-algebras on Ω generated, for $k \geqslant n$, by sets $\{\omega\colon x(k,\omega) = 0\}$ and $\{\omega\colon x(-k,\omega) = 0\}$, respectively. Let β denote the standard Bernoulli measure on Ω, i.e.

$$\beta\big(\{\omega\colon x(k,\omega) = \varepsilon_k,\ 1 \leqslant |k| \leqslant n\}\big) = \frac{1}{4^n} \quad \text{for all } n \geqslant 1 \text{ and } \{\varepsilon_k\} \subseteq \{0,1\}.$$

Take β_+ and β_- to be the restrictions of β to $\mathscr{M}^{(1)}$ and $\mathscr{N}^{(1)}$, respectively. Finally, put

$$\mathscr{J} = \bigcap_{1}^{\infty} \mathscr{M}^{(n)} \quad \text{and} \quad \mathscr{T} = \bigcap_{1}^{\infty} \mathscr{N}^{(n)},$$

and observe that, by Kolmogorov's 0-1 law, β_+ and β_- are degenerate on $\mathscr{J}$ and $\mathscr{T}$, respectively.

Next write

$$X = \{\omega \in \Omega\colon \lim_{n\to\infty} |x(n,\omega) - x(-n,\omega)| = 0\}.$$

Note that if $C \in \mathscr{M}^{(1)} \cup \mathscr{N}^{(1)}$ and $X \subseteq C$, then $\Omega = C$. Hence X has outer measure 1 with respect to both β_+ and β_-, and so there exist μ on $\langle X, \mathscr{A}\rangle$ and ν on $\langle X, \mathscr{B}\rangle$, $\mathscr{A} = \mathscr{M}^{(1)}[X]$ and $\mathscr{B} = \mathscr{N}^{(1)}[X]$, such that

$$\mu(M \cap X) = \beta_+(M) \quad \text{and} \quad \nu(N \cap X) = \beta_-(N) \qquad \text{for } M \in \mathscr{M}^{(1)} \text{ and } N \in \mathscr{N}^{(1)}.$$

Note that $\mathscr{D} \equiv \mathscr{I}[X] = \mathscr{T}[X]$ and that μ and ν are equal and degenerate on $\mathscr{D}$. Moreover, $\mathscr{A}$ and $\mathscr{B}$ are clearly conditionally independent, given $\mathscr{D}$.

Now suppose that η is a splicing of μ and ν. Clearly,

$$X = \bigcup_{n=1}^{\infty} \bigcap_{k \geqslant n} \{\omega\colon x(k, \omega) = x(-k, \omega)\}.$$

On the other hand,

$$\eta\big(\{\omega\colon x(k, \omega) = x(-k, \omega),\ n \leqslant k \leqslant N\} \cap X\big) = \frac{1}{2^{N-n+1}},$$

and so, if η is countably additive, then $\eta(X) = 0$. Hence there is no splicing of μ and ν.

Finally, we will show that μ and ν satisfy (2) and, therefore, that our example provides the second proof that (2) is not sufficient for the existence of a splicing. Suppose $A \in \mathscr{A}$, $B \in \mathscr{B}$, and $\mu(A)\nu(B) > 0$. We must show that $A \cap B \neq \emptyset$. Choose compact sets K and J in Ω so that $K \in \mathscr{M}^{(1)}$, $J \in \mathscr{N}^{(1)}$, $K \cap X \subseteq A$, $J \cap X \subseteq B$, and $\beta_+(K)\beta_-(J) > 0$. Clearly, it is enough to show that $J \cap K \cap X \neq \emptyset$. To this end, write

$$\tilde{K} = \{\omega \in \Omega\colon (\exists \omega' \in K) \lim_{n\to\infty} |x(n, \omega) - x(n, \omega')| = 0\},$$

$$\tilde{J} = \{\omega \in \Omega\colon (\exists \omega' \in J) \lim_{n\to\infty} |x(n, \omega) - x(-n, \omega')| = 0\}.$$

Then $\tilde{K} \in \mathscr{I}$ and $\tilde{J} \in \mathscr{T}$. Since $\beta_+(K)\beta_-(J) > 0$, we have $\beta_+(\tilde{K})\beta_-(\tilde{J}) > 0$, and so $\beta_+(\tilde{K}) = \beta_-(\tilde{J}) = 1$. Hence $\beta(\tilde{K} \cap \tilde{J}) = 1$. In particular, $\tilde{K} \cap \tilde{J} \neq \emptyset$; and, therefore,

$$(\exists \omega' \in K)(\exists \omega'' \in J) \lim_{n\to\infty} |x(n, \omega') - x(-n, \omega'')| = 0.$$

Define $\omega \in \Omega$ by $x(n, \omega) = x(n, \omega')$ and $x(-n, \omega) = x(-n, \omega'')$, $n \geqslant 1$. Then $\omega \in K \cap J \cap X$.

Remark 2. The following is typical of the sort of way in which conditionally independent σ-algebras arise in the study of stochastic processes. Let $X = D([0, 1], R)$ be the space of right-continuous functions on $[0, 1]$ having left limits. For $t \in [0, 1]$, let $x(t)$ be the evaluation map on X at time t. If $\mathscr{A}$ is the smallest σ-algebra with respect to which all $x(t)$, $t < \frac{1}{2}$, are measurable, and $\mathscr{B}$ is defined correspondingly for $x(t)$, $t \geqslant \frac{1}{2}$, then $\mathscr{A}$ and $\mathscr{B}$ are independent (i.e. $A \cap B = \emptyset \Rightarrow A = \emptyset$ or $B = \emptyset$), and so every μ on $\mathscr{A}$ and ν on $\mathscr{B}$ admit a splicing (simply take $\mathscr{D} = \{\emptyset, X\}$ in Corollary 4). If $\mathscr{A}$ is generated by the $x(t)$ for $t \leqslant \frac{1}{2}$ and $\mathscr{B}$ is as before, then $\mathscr{A}$ and $\mathscr{B}$ are conditionally independent, given the σ-algebra $\mathscr{D}$ with respect to which $x(\frac{1}{2})$ is measurable. Hence μ and ν

admit a splicing in this case if and only if

$$(\exists a \in R)\ \mu\big(x(\tfrac{1}{2}) = a\big) = \nu\big(x(\tfrac{1}{2}) = a\big) = 1.$$

See Lemma 3.6 of [5] for the use of this sort of result.

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Reçu par la Rédaction le 2. 6. 1974