COLLOQUIUM MATHEMATICUM

VOL. XLI 1979 FASC. 2

OPERATOR-STABLE PROBABILITY MEASURES ON BANACH SPACES

BY

W. KRAKOWIAK (WROCŁAW)

Introduction. An *operator-stable probability measure* in a real separable Banach space X is a limit law arising, roughly speaking, from affine modification of the partial sums of a sequence of independent identically distributed X-valued random variables. We show that a probability measure μ is operator-stable if and only if, for each $t > 0$, μ^t is a translation of the measure $e^{\log tB}\mu$ for any linear continuous operator B on X. Further, we get a representation of the characteristic functionals of these limit laws.

1. Notation and preliminaries. Let X denote a real separable Banach space with the norm $\|\cdot\|$ and with the dual space X^*. By $\langle\cdot,\cdot\rangle$ we shall denote the dual pairing between X and X^*. Further, $B(X)$ will denote the algebra of continuous, linear operators on X with the norm topology. Given a subset F of $B(X)$, by $\mathrm{Sem}(F)$ we shall denote the closed multiplicative semigroup of operators spanned by F. The unit and the zero operators will be denoted by I and 0, respectively. A sequence $\{\mu_n\}$ of probability measures on X is said to *converge to a probability measure* μ on X if, for every bounded continuous real-valued function f on X,

$$\int_X f d\mu_n \to \int_X f d\mu.$$

The characteristic functional of μ is defined on X^* by the formula

$$\hat{\mu}(y) = \int_X e^{i\langle x,y\rangle}\mu(dx), \text{ where } y \in X^*.$$

For an operator A from $B(X)$ and a probability measure μ on X let $A\mu$ denote the probability measure defined by the formula $A\mu(E)$

$= \mu(A^{-1}(E))$ for all Borel subsets E of X. It is easy to check the equations

$$A(\mu * \nu) = A\mu * A\nu \quad \text{and} \quad \widehat{A\mu}(y) = \hat{\mu}(A^* y),$$

where A^* denotes the adjoint operator.

Moreover, $A_n\mu_n \to A\mu$ whenever $A_n \to A$ and $\mu_n \to \mu$. Given a probability measure μ on X, we define $\overline{\mu}$ by putting $\overline{\mu}(E) = \mu(-E)$, where $-E = \{-x\colon x \in E\}$. For any probability measure μ on X the measure $|\mu|^2 = \mu * \overline{\mu}$ is called the *symmetrization* of μ. A probability measure μ is said to be *full* if its support is not contained in any proper hyperplane of X. Moreover, by δ_x $(x \in X)$ we denote the probability measure concentrated at the point x.

A probability measure μ on X is said to be *infinitely divisible* whenever for every positive integer n there exists a probability measure μ_n such that $\mu = \mu_n^{*n}$, where the power is taken in the sense of convolution. For the theory of infinitely divisible probability measures on Banach spaces and on even more general algebraic structures, we refer the reader to [15], [16] and [3]. In particular, if F is any bounded non-negative Borel measure, then $e(F)$ associated with F is defined by

$$e(F) = e^{-F(x)} \sum_{k=0}^{\infty} \frac{1}{k!} F^{*k},$$

where $F^{*0} = \delta_0$. The measure F is called a *Poisson exponent* of $e(F)$. Let M be a not necessarily bounded Borel measure on X vanishing at $\{0\}$. If there exists a representation $M = \sup F_n$, where the F_n's are bounded and the sequence $\{e(F_n)\}$ of associated Poisson measures is shift compact, then each cluster point of the sequence $\{e(F_n) * \delta_{x_n}\}$ $(x_n \in X)$ is called a *generalized Poisson measure* and denoted by $\tilde{e}(M)$. Clearly, $\tilde{e}(M)$ is uniquely determined up to translation, i.e., for two cluster points, say μ_1 and μ_2 of $\{e(F_n) * \delta_{x_n}\}$ and $\{e(F_n) * \delta_{y_n}\}$, respectively, we have $\mu_1 = \mu_2 * \delta_x$ for certain $x \in X$ ([15], p. 313). Further, the measure M is called a *generalized Poisson exponent* of $\tilde{e}(M)$. Let $M(X)$ denote the set of all generalized Poisson exponents of X.

By a *Gaussian measure* on X we mean such a probability measure ϱ on X that for every $y \in X^*$ the induced measure $y\varrho$ on the real line is Gaussian. Gaussian measures on Banach spaces have been studied by Fernique in [4], Kuelbs in [8], and Vakhania in [18]. In this paper we consider only symmetric Gaussian measures. For such measures the characteristic functional is of the form

$$\hat{\varrho}(y) = \exp[-\tfrac{1}{2}\langle y, Ry\rangle] \quad (y \in X^*),$$

where R is the covariance operator, i.e., a compact operator from X^* into X with the properties: $\langle y_1, Ry_2\rangle = \langle y_2, Ry_1\rangle$ for all $y_1, y_2 \in X^*$ (symmetry) and $\langle y, Ry\rangle \geqslant 0$ (non-negativity) ([18], p. 136, [2]). By

$R(X)$ we shall denote the set of all covariance operators of Gaussian measures on X. Clearly, if R is the covariance operator of ϱ and $A \in B(X)$, then ARA^* is the covariance operator of $A\varrho$.

Tortrat proved in [15], p. 311 (see also [2]), the following analogue of the Lévy-Khinchine representation of infinitely divisible laws.

PROPOSITION 1.1. *A probability measure μ on X is infinitely divisible if and only if*

$$\mu = \varrho * \tilde{e}(M), \tag{1.1}$$

where ϱ is a symmetric Gaussian measure and $M \in M(X)$. Moreover, decomposition (1.1) *is unique.*

Let μ be an infinitely divisible probability measure on X. Then for every $c \geqslant 0$ there exists an infinitely divisible probability measure ν on X such that $\hat{\nu}(y) = [\hat{\mu}(y)]^c$. We denote ν by μ^c. The set $\{\mu^c\}_{c\geqslant 0}$ is an abelian semigroup with the convolution as a semigroup operation, and the mapping $c \to \mu^c$ is a homomorphism of the additive semigroup of non-negative real numbers onto $\{\mu^c\}_{c\geqslant 0}$. Moreover, the mapping $c \to \mu^c$ is continuous. Namely, we prove the following

PROPOSITION 1.2. *Let μ be an infinitely divisible probability measure on X and let $\{c_n\}$ be a sequence of non-negative real numbers converging to c_0. Then μ^{c_n} converges to μ^{c_0}.*

Proof. Let $\mu = \varrho * \tilde{e}(M)$, where ϱ is a symmetric Gaussian measure and $M \in M(X)$. Hence $\mu^{c_n} = \varrho^{c_n} * \tilde{e}(M)^{c_n}$. Since the sequence $\{c_n\}$ is bounded, the sequence $\{\mu^{c_n}\}$ is shift compact (Theorem 3.2.2 of [12]). Further, the sequence $\{\varrho^{c_n}\}$ is conditionally compact and $\{\tilde{e}(M)^{c_n}\}$ is shift compact. Thus ϱ^{c_n} converges to ϱ^{c_0}. Without loss of generality we may assume that $c_0 = 0$. Let $\nu = \tilde{e}(M)$ and $\nu_n = \tilde{e}(M)^{c_n}$. We now show that the sequence ν_n converges to δ_0. By Lemma 1.2.4 from [3], this is equivalent to

$$\lim_{n\to\infty} \sup_{y\in U_r^0} |\hat{\nu}_n(y)-1| = 0 \quad \text{or} \quad \lim_{n\to\infty} \sup_{y\in U_r^0} |\log\hat{\nu}_n(y)| = 0$$

for all $r > 0$, where $U_r = \{x \in X\colon \|x\| \leqslant r\}$ and

$$U_r^0 = \{y \in X^*\colon |\langle x, y\rangle| \leqslant 1 \text{ for all } x \in U_r\}.$$

There exists a $\delta > 0$ such that $|t-\sin t| \leqslant 1-\cos t$ for $|t| \leqslant \delta$. From the Dettweiler representation of the characteristic functionals of infinitely divisible measures on X (Theorem 1.2.5 of [3]) we get the formula

$$\hat{\nu}(y) = \exp\Big\{i\langle a, y\rangle + \int_X [e^{i\langle x,y\rangle} - 1 - i\langle x, y\rangle 1_{U_{\delta_r}}(x)]\, M(dx)\Big\},$$

where $a \in X$ and $1_{U_{\delta_r}}$ denotes the indicator of U_{δ_r}. Hence

$$\log\hat{\nu}_n(y) = i\langle c_n a, y\rangle + \int_X [e^{i\langle x,y\rangle} - 1 - i\langle x, y\rangle 1_{U_{\delta_r}}(x)]\, c_n M(dx).$$

If $y \in U_r^0$, then $|\langle x, y\rangle| \leqslant \delta$ for all $x \in U_{\delta_r}$. Further, we get

$$|\log \hat{\nu}_n(y)| \leqslant Kc_n + c_n \int_{U_{\delta_r}} |e^{i\langle x,y\rangle} - 1 - i\langle x, y\rangle| M(dx)$$
$$\leqslant Kc_n + c_n \int_X (1 - \cos\langle x, y\rangle) 2M(dx) = Kc_n + \log |\hat{\nu}_n(y)|^2.$$

Let $\varepsilon > 0$. Then there exists an n_0 such that $Kc_n \leqslant \varepsilon/2$ for $n \geqslant n_0$. Since $|\tilde{e}(M)^{c_n}|^2$ converges to δ_0, we now conclude by Lemma 1.2.3 of [3] that there exists an n such that

$$\sup_{y \in U_r^0} \log |\hat{\nu}_n(y)|^2 \leqslant \frac{\varepsilon}{2} \quad \text{for all } n \geqslant n_1.$$

Let $n_2 = \max(n_0, n_1)$. We have

$$\sup_{y \in U_r^0} |\log \hat{\nu}_n(y)| < \varepsilon \quad \text{for all } n \geqslant n_2.$$

This completes the proof of Proposition 1.2.

The following lemma will be used repeatedly in the sequel.

LEMMA 1.1. *Let μ, ν, γ_n $(n = 1, 2, \ldots)$ be probability measures on X such that*

$$\lim_{n\to\infty} \hat{\mu}(y)\hat{\gamma}_n(y) = \hat{\nu}(y) \quad \textit{for all } y \in X^*.$$

Assume that $\hat{\nu}(y) \neq 0$ for all $y \in X^*$. Then there exists a unique probability measure γ on X such that

$$\mu * \gamma = \nu. \tag{1.2}$$

Proof. Let $f(y) = \hat{\nu}(y)/\hat{\mu}(y)$ for all $y \in X^*$. The function f is continuous. Let $\mathbf{N}$ be a collection of finite codimension and closed subspaces of X and let $p_N\colon X \to X/N$ $(N \in \mathbf{N})$ be canonical maps. For all $N \in \mathbf{N}$ we have

$$\lim_{n\to\infty} p_N\gamma_n(y) = f(p_N^*(y)), \quad y \in (X/N)^*.$$

Thus there exists a unique probability measure γ_N on X/N such that the sequence $\{p_N\gamma_n\}$ converges to γ_N and $\hat{\gamma}_N = fp_N^*$. Further, we have

$$p_N(\mu) * \gamma_N = p_N(\nu). \tag{1.3}$$

The system of measure $\{\gamma_N\}_{N\in\mathbf{N}}$ is a cylindrical measure. By (1.3) and Lemma 1.1.7 of [3] there exists a probability measure γ on X such that (1.2) holds, which completes the proof.

2. Statement of the problem. In terms of random variables, the problem we study is enunciated as follows:

Suppose that $\{\xi_n\}$ is a sequence of independent identically distributed X-valued random variables and assume that $\{A_n\}$ and $\{x_n\}$ are sequences from $B(X)$ and X, respectively, such that

(*) A_n are invertible,

(**) $\mathrm{Sem}(\{A_m A_n^{-1}\colon n = 1, 2, \ldots, m; m = 1, 2, \ldots\}) = \mathcal{S}$ is compact (in the norm topology of $B(X)$),

(***) the distribution of

$$A_n \sum_{j=1}^{n} \xi_j + x_n$$

converges to a probability measure μ.

What can be said about the limit measure μ?

In the one-dimensional case this problem has been solved by P. Lévy: the class of all limit measures in question coincides with the class of all stable probability measures (see [11], p. 326). Therefore, the limit measures μ will be called *operator-stable measures*. Evidently, operator-stable measures are Lévy's measures (see [17]). This paper, stimulated by results of Urbanik [17], is an outgrowth of Sharpe's work [14] concerning operator-stable measures on finite-dimensional spaces. All that has been done so far for Banach spaces describes the limit measures when all operators A_n are multiples of the unit operator. In this case, Kumar and Mandrekar proved in [10] an analogue of the Lévy characterization theorem, and Jurek and Urbanik obtained in [7] a representation of the characteristic functional.

We note that for full operator-stable measures on finite-dimensional spaces the compactness condition (**) can be omitted. The same is true for non-degenerate measures on a Banach space when A_n are multiples of I.

3. Characterization of full operator-stable measures. We say that a sequence $\{A_n\}$ of operators from $B(X)$ with properties (*) and (**) is a *norming sequence* corresponding to an operator-stable measure μ if there exist a probability measure ν on X and a sequence $\{a_n\}$ of elements of X such that $A_n \nu^{*n} * \delta_{a_n}$ converges to μ.

LEMMA 3.1. *Let μ be a full operator-stable measure on X and let $\{A_n\}$ be a norming sequence corresponding to μ. Then for every $c \in (0,1)$ there exist $B_c \in \mathcal{S}$ and $a_c \in X$ such that*

$$\mu^c = B_c \mu * \delta_{a_c}. \tag{3.1}$$

Proof. Let $c \in (0,1)$ and let $\{n_k\}$ be a sequence of positive integers such that

$$\lim_{k\to\infty} \frac{n_k}{n_{k+1}} = c.$$

We have

$$A_{n_{k+1}}\nu^{*n_{k+1}} * \delta_{a_{n_{k+1}}} \to \mu \quad \text{as } k \to \infty.$$

Hence

$$[\hat{\nu}(A^*_{n_{k+1}}y)]^{n_{k+1}} \exp[i\langle a_{n_{k+1}}, y\rangle] \to \hat{\mu}(y) \quad \text{for all } y \in X^*.$$

Let

$$b_k = \frac{n_k}{n_{k+1}} a_{n_{k+1}} - A_{n_{k+1}} A^{-1}_{n_k}(a_{n_k}).$$

Further, the sequence

$$\{[\hat{\nu}(A^*_{n_k}(A_{n_{k+1}}A^{-1}_{n_k})^*(y))]^{n_k} \exp[i\langle A_{n_{k+1}}A^{-1}_{n_k}(a_{n_k}), y\rangle]\exp[i\langle b_k, y\rangle]\}$$

converges to $[\hat{\mu}(y)]^c$ for all $y \in X^*$. Since the sequence $\{A_{n_{k+1}}A^{-1}_{n_k}\}$ is conditionally compact, we may assume without loss of generality that

$$A_{n_{k+1}}A^{-1}_{n_k} \to B_c. \tag{3.2}$$

By (3.2) we have

$$(A_{n_{k+1}}A^{-1}_{n_k})(A_{n_k}\nu^{*n_k} * \delta_{a_n}) \to B_c\mu.$$

Clearly,

$$(\hat{\mu}B^*_c y)\exp[i\langle b_k, y\rangle] \to [\hat{\mu}(y)]^c \quad \text{for all } y \in X^*.$$

By Lemma 1.2, (3.1) holds. This completes the proof of the lemma.

LEMMA 3.2. *Let μ be a full operator-stable measure on X and let $\{A_n\}$ be a norming sequence corresponding to μ. Then there exist sequences $\{B_n\}$ and $\{c_n\}$ of elements of $\mathcal{S}$ and of elements of $(0, 1)$, respectively, such that $B_n \to I$, $c_n \to 1$ and*

$$\mu^{c_n} = B_n\mu * \delta_{b_n} \quad (n = 1, 2, \ldots), \tag{3.3}$$

where $b_n \in X$.

Proof. Let $\{c_n\}$ be any sequence of elements of $(0, 1)$ such that $c_n \to 1$. By Lemma 3.1, there exists a sequence $\{C_n\}$, $C_n \in \mathcal{S}$, such that $\mu^{c_n} = C_n\mu * \delta_{a_n}$ $(a_n \in X)$ for all n. Since $\{C_n\}$ is conditionally compact, we may assume without loss of generality that $C_n \to C$. We have $\mu^{c_n} \to \mu$ (Proposition 1.1) and $C_n\mu \to C\mu$, whence, by Lemma 1.1, $\mu = B\mu * \delta_b$ for certain $b \in X$. Further, by Lemma 3.2 of [17], the operator B is invertible. Let $B_n = C_nC^{-1}$. Now we have (3.3), which completes the proof of the lemma.

Now, we are ready to prove a characterization theorem for a full operator-stable measure on X.

THEOREM 3.1. *A full probability measure μ on a real separable Banach space X is an operator-stable measure if and only if there is an operator $B \in B(X)$ with*

$$\lim_{t\to 0} \exp[\log tB] = 0$$

such that

$$\mu^t = \exp[\log tB]\mu * \delta_{b_t}, \quad t > 0, \tag{3.4}$$

where $b_t \in X$.

Proof. Necessity. Suppose that μ is an operator-stable measure and $\{A_n\}$ is a norming sequence corresponding to μ. By Lemma 3.2 there exist sequences $\{B_n\}$ and $\{c_n\}$ of operators of $\mathscr{S}$ and of real numbers of $(0, 1)$, respectively, such that $B_n \to I$, $c_n \to 1$ and $\mu^{c_n} = B_n\mu * \delta_{b_n}$ $(b_n \in X)$ for $n = 1, 2, \ldots$

Let W be the set of all rational numbers of $(0, 1)$ and let $w \in W$. Then there exists a sequence $\{k_n^w\}$ of positive integers such that

$$c_n^{z+1} < w \leqslant c_n^z$$

and

$$\lim_{n\to\infty} c_n^z = w, \tag{3.5}$$

where z stands for k_n^w.

Further, we get

$$\mu^{c_n^z} = B_n^z\mu * \delta_{a_n}$$

for certain $a_n \in X$ and $n = 1, 2, \ldots$ Since, by Proposition 1.1 and (3.5), $\mu^{c_n^z} \to \mu^w$ and the sequence $\{B_n^z\}$ is conditionally compact, we have

$$\mu^w = Q_w\mu * \delta_{b_w}$$

for certain $b_w \in X$ and $w \in W$.

The set $\{Q_w\}_{w\in W}$ is conditionally compact and

$$\lim_{w\to 0} Q_w = 0.$$

We may assume without loss of generality that

$$Q_{w_1}Q_{w_2} = Q_{w_2}Q_{w_1}, \quad w_1, w_2 \in W,$$

and

$$Q_{w_1}Q_{w_2} = Q_{w_1w_2}. \tag{3.6}$$

Now we prove that operators from $\{Q_w\}_{w\in W}$ are invertible.

By (3.6) it is enough to show that there exists an ε $(0 < \varepsilon < 1)$ such that the operators Q_w $(1-\varepsilon < w < 1)$ are invertible. Suppose that $\{w_n\}$ and $\{Q_{w_n}\}$ are sequences of rational numbers of $(0, 1)$ and of operators of $\{Q_w\}_{w\in W}$, respectively, such that $w_n \to 1$, Q_{w_n} are non-invertible and $\mu^{w_n} = W_{w_n}\mu * \delta_{b_n}$ for certain $b_n \in X$. We may assume without loss of generality that $Q_{w_n} \to Q$. By Lemma 3.2 of [17] the operator Q is invertible, which is a contradiction since the set of invertible operators is open in the norm topology of $B(X)$.

The set $\{Q_w\}_{w\in W}\cup\{I\}\cup\{Q_w^{-1}\}_{w\in W}$ is a group. Let H be its closure in the norm topology of $B(X)$. We write

$$G_p = \{A\in B(X)\colon \mu^p = A\mu * \delta_a,\ a\in X\}\cap H, \quad 0<p<\infty.$$

Clearly,

$$G_p\cap G_q = \emptyset \quad \text{if } p\neq q \tag{3.7}$$

and

$$H = \bigcup_{0<p<\infty} G_p\cup\{0\}.$$

G_1 is an abelian compact group. Let F be a set containing all cluster points of $\{Q_w\}_{w\in W}$. Then the set $F\cap G_1$ is an abelian compact group (Lemma 3.2 of [17]) and $G_1 = G_1\cap F$.

Operators from $\bigcup_{0<p<\infty} G_p$ are invertible. Clearly, there exists an $s\in(0,1)$ such that the operators from $\bigcup_{s<p\leqslant 1} G_p$ are invertible. Let $0<p<s$ and $A\in G_p$. Then there exist sequences $\{w_n\}$ and $\{Q_{w_n}\}$ of rational numbers of $(0,1)$ and of operators of $\{Q_w\}_{w\in W}$, respectively, such that

$$\lim_{n\to\infty} w_n = p \quad \text{and} \quad \lim_{n\to\infty} Q_{w_n} = A.$$

Let $v = \frac{1}{2}(1+s)$ and $v_n = w_n v p^{-1}$. We may assume without loss of generality that $v_n<1$ for $n = 1, 2, \ldots$ We have

$$Q_{w_n}Q_vQ_p^{-1} = Q_{w_n v}Q_p^{-1} = Q_{w_n v p^{-1}}Q_pQ_p^{-1} = Q_{v_n}$$

and Q_{v_n} converges to $AQ_vQ_p^{-1}$. Since $AQ_vQ_p^{-1}\in G_v$, $AQ_vQ_p^{-1}$ is invertible, and so is A. Thus operators from $\bigcup_{0<p\leqslant 1} G_p$ are invertible.

Further, we get

$$G_p = G_{1/p}^{-1} \quad \text{for } p\in(0,\infty)$$

and

$$G_pG_q = G_{pq} \quad \text{for } p, q\in(0,\infty). \tag{3.8}$$

Clearly, for any $0<r\leqslant q<\infty$ the set $\bigcup_{r\leqslant p\leqslant q} G_p$ is compact.

Let

$$G = \bigcup_{0<p<\infty} G_p.$$

Then G is a locally compact, compactly generated, abelian group. G_1 is a compact maximal subgroup of G.

Let f be a function of G into the positive real numbers such that $f(Q) = p$ if $Q\in G_p$. By (3.7) the mapping f is well defined. In view of (3.8), f is a continuous homomorphism. The openness of f is now a consequence of its continuity and of the σ-compactness of G (Theorem 5.29 of [5]). G_1 is the kernel of f.

The mapping $\log f\colon G \to R$ is a continuous open homomorphism of G onto the additive group R. The kernel of $\log f$ is G_1. By Theorem 5.27 of [5], G/G_1 is isomorphic to R. From the Pontrjagin theorem ([19], § 29) we infer that G is isomorphic to the direct sum of R and G_1. Let $g\colon R \times G_1 \to G$ be such an isomorphism and let $W_t = g(\langle t, I\rangle)$. It is clear that $\{W_t\}_{t\in R}$ is a continuous one-parameter group of operators from G satisfying the condition

$$\lim_{t\to 0} W_t = I.$$

By Theorem 8.4.2 of [6], it can be represented in the exponential form $W_t = \exp[tB]$, where $B \in B(X)$. Moreover, $W_t \notin G$ for $t \neq 0$.

The mapping $t \to \exp[tB]$ is a continuous homomorphism of the additive group R into G, so

$$t \to \exp[tB] \to \log f(\exp[tB])$$

is a continuous homomorphism of the additive group R onto R such that $\log f(\exp[tB]) = Kt$ for some constant $K \in R$. Replacing B by $K^{-1}B$, we may assume that $\log f(\exp[tB]) = t$, giving

$$f(\exp[tB]) = \exp[t] \quad \text{or} \quad f(\exp[\log tB]) = t.$$

Thus $\exp[\log tB] \in G_t$ for every $t > 0$.

Sufficiency. Assuming that there exists an operator $B \in B(X)$ with properties as in Theorem 3.1 and taking $A_n = \exp[\log n^{-1}B]$, we obtain

$$\mu = A_n \mu^n * \delta_{nb_{1/n}}$$

and

$$\mathscr{S} = \{\exp[\log tB]\colon 0 < t \leqslant 1\} \cup \{0\}.$$

Thus the theorem is proved.

Remark 3.1. We shall denote $\exp[\log tB]$ by t^B.

COROLLARY 3.1. *A full probability measure on a real separable Banach space X is operator-stable if and only if there exists a sequence $\{A_n\}$ of invertible operators of $B(X)$ such that $\mathscr{S}$ is compact in the norm topology of $B(X)$ and*

$$\mu = A_n \mu^{*n} * \delta_{a_n} \quad (n = 1, 2, \ldots)$$

for some $a_n \in X$.

Remark 3.2. Let G be a group of invertible operators of $B(X)$. A probability measure μ on X is called *stable* under G if for any $A, B \in G$ there exist $C \in A$ and $x \in X$ such that

$$A\mu * B\mu = C\mu * \delta_x.$$

Let

$$A \in B(X) \quad \text{with} \lim_{t\to -\infty} \exp[tA] = 0.$$

A full probability measure μ is stable under $\{\exp[tA]\}_{t\in R}$ if and only if μ is an operator-stable measure and $\mu^t = t^B\mu * \delta_{b_t}$, where $t > 0$, $b_t \in X$ and $B = cA$ for some $c > 0$ (see [13]).

4. Representation of operator-stable measures. Our next aim is to give a representation of the characteristic functionals of operator-stable measures on X.

THEOREM 4.1. *Let*

$$B \in B(X) \quad \textit{with} \lim_{t\to 0} t^B = 0.$$

Then a full probability measure μ is operator-stable with $\mu^t = t^B\mu * \delta_{b_t}$ for all $t > 0$ and for some $b_t \in X$ if and only if $\mu = g * \tilde{e}(M)$, where g is a symmetric Gaussian measure with the covariance operator R and $M \in M(X)$ such that $R = BR + RB^*$ and $t^BM = tM$ for all $t > 0$.

Proof. Necessity. Suppose that μ is operator-stable and has the properties as in the theorem. Since μ is infinitely divisible, $\mu = g * \tilde{e}(M)$, where g is a symmetric Gaussian measure with the covariance operator R and $M \in M(X)$. Moreover, for all $t > 0$ we have

$$t^BM = tM, \quad TR = t^BRt^{B*}.$$

By a simple calculation we get the formulas

$$\lim_{t\to 0} \frac{R-(\exp[tB])R(\exp[tB^*])}{t} = -(BR+RB^*)$$

and

$$\lim_{t\to 0} \frac{1-e^t}{t} R = -R,$$

which implies the equation $R = BR+RB^*$.

Sufficiency. Let us assume that $M \in M(X)$, $t^BM = tM$, $R \in R(X)$, and $R = BR+RB^*$. Clearly,

$$\tilde{e}(M)^t = t^B\tilde{e}(M).$$

Let $A(t) = e^tR-(\exp[tB])R(\exp[tB^*])$ for all $t > 0$. Clearly, for all $t > 0$

$$\lim_{t\to 0} \frac{A(t+h)-A(t)}{h} = A(t).$$

Given $y_1, y_2 \in X^*$, we put $f_{y_1,y_2}(t) = \langle y_1, A(t)y_2\rangle$. Further, we get

$$f_{y_1,y_2}(0) = 0, \quad \frac{d}{dt} f_{y_1,y_2}(t) = f_{y_1,y_2}(t).$$

Evidently, for all $y_1, y_2 \in X^*$ and $t \in R$ we have

$$\langle y_1, A(t)y_2\rangle = 0.$$

Thus $A(t) = 0$, which completes the proof.

COROLLARY 4.1. *Let*

$$B \in B(X) \quad \textit{with } \lim_{t\to 0} t^B = 0$$

and let μ *be a full operator-stable measure on* X *with* $\mu^t = t^B\mu * \delta_{b_t}$ *for all* $t > 0$ *and for some* $b_t \in X$. *If* $\mu = g * \tilde{e}(M)$, *where* g *is a symmetric Gaussian measure and* $M \in M(X)$, *then* g *and* M *are concentrated on subspaces* X_1 *and* X_2, *respectively, which are invariant under* B *and* $X = X_1 + X_2$.

Let $B \in B(X)$ with $\lim_{t\to 0} t^B = 0$. Given a subset E of X, we put

$$\tau(E) = \{t^B x\colon x \in E, 0 < t < \infty\}.$$

It is clear that for any compact set E with the property $0 \notin E$ and for any pair r_1, r_2 $(r_1 > r_2)$ of positive numbers the inequality

$$r_1 \leqslant \|t_n^B x_n\| \leqslant r_2 \qquad (x_n \in E)$$

implies the existence of certain pair c_1, c_2 $(c_1 < c_2)$ of positive numbers such that $c_1 \leqslant t_n \leqslant c_2$ for $n = 1, 2, \ldots$ This simple fact yields the following

LEMMA 4.1. *Let* E *be a compact subset of* X *and let* $0 \notin E$. *Then for every pair* r_1, r_2 $(r_1 \leqslant r_2)$ *of positive numbers the set* $\{x\colon r_1 \leqslant \|x\| \leqslant r_2\} \cap \tau(E)$ *is compact.*

The following lemma reduces our problem of examining measures $M \in M(X)$ with the property $t^B M = tM$ $(t > 0)$ to the case of measures concentrated on $\tau(E)$, where E is compact and $0 \notin E$.

LEMMA 4.2. *Let* $M \in M(X)$ *and* $tM = t^B M$ *for all* $t > 0$. *Then there exists a decomposition*

$$M = \sum_{n=1}^{\infty} M_n,$$

where $M_n \in M(X)$, $tM_n = t^B M_n$ *for all* $t > 0$, M_n *are concentrated on disjoint sets* $\tau(E_n)$, $0 \notin E_n$, *and* E_n *are compact.*

The proof of the lemma is immediate by Lemma 5.4 of [17].

Now, we are ready to prove the representation of the characteristic functionals of full operator-stable measures.

THEOREM 4.2. *A full probability measure* μ *on* X *is an operator-stable measure if and only if there exist an operator* $B \in B(X)$ *with* $\lim_{t\to 0} t^B = 0$, *an element* $a \in X$, *an operator* $R \in R(X)$ *such that* $R = BR + RB^*$, *and* *a finite measure γ on the unit sphere S of X such that*

$$\hat{\mu}(y) = \exp\Big\{i\langle a, y\rangle - \tfrac{1}{2}\langle y, Ry\rangle + \\ + \int_S \int_0^\infty \big[\exp[i\langle t^B x, y\rangle] - 1 - i\langle t^B x, y\rangle 1_D(t^B x)\big] t^{-2} dt \gamma(dx)\Big\}, \tag{4.1}$$

where 1_D denotes the indicator of the unit ball D in X and $y \in X^*$.

Proof. We use arguments similar to those given by Kuelbs in [9]. Let μ be a full operator-stable measure on X. Hence μ is infinitely divisible and $\mu = \varrho * \tilde{e}(M)$, where ϱ is a symmetric Gaussian measure with the covariance operator R and $M \in M(X)$. Moreover, for all $t > 0$ we have

$$t^B M = tM, \quad R = BR + RB^*.$$

Let M be a decomposition as in Lemma 4.2 and let

$$D_n = \tau(E_n) \cap \{x: \|x\| = 1\}.$$

By Lemma 4.1 the set D_n is compact. We define an equivalence relation in D_n as follows:

$x_1 \sim x_2$, $x_1, x_2 \in D_n$, if and only if there exists a $t > 0$ such that $x_1 = t^B x_2$.

In order to prove the continuity of this equivalence relation suppose that $x_n \sim x_n^1$ and that the sequences $\{x_n\}$ and $\{x_n^1\}$ converge to x and x^1, respectively. Then for some real positive numbers t_n we have $t_n^B x_n = x_n^1$. From the compactness of E_n and the assumption $0 \notin E_n$ we infer that there exists a certain pair c_1, c_2 $(c_1 < c_2)$ of positive numbers such that $c_1 \leqslant t_n \leqslant c_2$ for $n = 1, 2, \ldots$ Clearly, for any cluster point t_0 of $\{t_n\}$ we have $t_0^B x \sim x^1$, which implies $x \sim x^1$. Thus the relation $\sim$ is continuous. Hence it follows that the quotient space $D_n/\sim$ is compact ([1], p. 97). The coset containing x will be denoted by $[x]$. Further, the mapping $x \to [x]$ from D_n onto $D_n/\sim$ is continuous. A theorem of Kuratowski (Theorem 1.4.2 of [12]) shows that there exists a Borel subset S_n of D_n such that S_n intersects each $[x]$ at exactly one point.

Let f_n be a mapping of $S_n \times (0, \infty)$ into $\tau(E_n)$ such that $f_n(x, t) = t^B x$. The mapping f_n is continuous and one-to-one. By another theorem of Kuratowski (Corollary 1.3.3 of [12]), the mapping f_n^{-1} is measurable. Let

$$f: \bigcup_{n=1}^{\infty} S_n \times (0, \infty) \xrightarrow{\text{into}} \bigcup_{n=1}^{\infty} \tau(E_n)$$

be such that $f(x, t) = f_n(x, t)$ if $x \in S_n$. Then f is one-to-one and the mappings f and f^{-1} are measurable. Hence the σ-field generated by the collection of the sets $\{t^B x: x \in I, x \in F\}$, where I and F are closed intervals on the half line $(0, \infty)$ and on the Borel set of $S_0 = \bigcup_{n=1}^{\infty} S_n$, respectively, consists of all Borel subsets of $\bigcup_{n=1}^{\infty} \tau(E_n)$.

Put

$$g(r, F) = M(\{t^B x\colon\ t \geqslant r,\, x \in F\}), \qquad r > 0.$$

Since $tM = t^B M$ for all $t > 0$, we have

$$g\left(\frac{r}{s}, F\right) = sg(r, F), \qquad r, s > 0.$$

Now, setting $r = s$ and $\gamma_0(F) = g(1, F)$ we get $g(s, F) = s^{-1}\gamma_0(F)$, which implies the formula

$$M(\{t^B x\colon\ t \in I,\, x \in F\}) = \gamma_0(F) \int_I t^{-2}\,dt. \tag{4.2}$$

Since there exists an $r > 0$ such that $\|t^B x\| \geqslant r$ for all $t \geqslant 1$ and $x \in S_0$, the measure γ_0 is finite.

Formula (4.2) can be extended for all Borel subsets of $X \setminus \{0\}$ as

$$M(F) = \int_S \int_0^{\infty} 1_F(t^B x) t^{-2}\,dt\,\gamma(dx), \tag{4.3}$$

where $\gamma(G) = \gamma_0(G \cap S_0)$ for any Borel subset G of S. Further, from the Dettweiler representation of the characteristic functionals of an infinitely divisible measure on X (Theorem 1.2.5 of [3]) we get the formula

$$\hat{\mu}(y) = \exp\Big\{ i\langle a, y\rangle - \tfrac{1}{2}\langle y, Ry\rangle + \int_X [e^{i\langle x, y\rangle} - 1 - i\langle x, y\rangle 1_D(x)] M(dx) \Big\}, \tag{4.4}$$

where $y \in X^*$, $a \in X$, $R \in R(X)$, $M \in M(X)$, and 1_D denotes the indicator of the unit ball D in X. Setting the expression (4.3) for M into (4.4) we get the required representation (4.1), which completes the proof of the necessity.

By a simple calculation one can check that each measure μ with the characteristic functional of form (4.1) satisfies equation (3.4) for all $t > 0$. Hence, by Theorem 3.1, we get the proof of the theorem.

REFERENCES

[1] N. Bourbaki, *Eléments de mathématique, I. Les structures fondamentales de l'analyse*, Livre III, *Topologie générale*, Paris 1951.

[2] С. А. Чобанян и В. И. Тариеладзе, *О полной непрерывности ковариационного оператора*, Сообщения Академии наук грузинской ССР 70 (1973), p. 273-276.

[3] E. Dettweiler, *Grenzwertsätze für Wahrscheinlichkeitsmasse auf Badrikianschen Räumen*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 34 (1976), p. 285-311.

[4] X. Fernique, *Intégrabilité des vecteurs gaussiens*, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Série A, 270 (1970), p. 1698-1699.

[5] E. Hewitt and K. Ross, *Abstract harmonic analysis*, Vol. I, Berlin 1963.

[6] E. Hille, *Functional analysis and semigroups*, New York 1948.

[7] Z. Jurek and K. Urbanik, *Remarks on stable measures on Banach spaces*, Colloquium Mathematicum 38 (1978), p. 269-276.

[8] J. Kuelbs, *Gaussian measures on a Banach space*, Journal of Functional Analysis 5 (1970), p. 354-367.

[9] — *A representation theorem for symmetric stable processes and stable measures on H*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 26 (1973), p. 259-271.

[10] A. Kumar and V. Mandrekar, *Stable probability measures on Banach spaces*, Studia Mathematica 42 (1972), p. 133-144.

[11] M. Loève, *Probability theory*, 3rd ed., New York 1963.

[12] K. S. Parthasarathy, *Probability measures in metric spaces*, New York 1967.

[13] K. Schmidt, *Stable probability measures on R^v*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 33 (1975), p. 19-31.

[14] M. Sharpe, *Operator-stable probability distributions on vector groups*, Transactions of the American Mathematical Society 136 (1969), p. 51-65.

[15] A. Tortrat, *Structure des lois indéfiniment divisibles dans un espace vectoriel topologique (séparé) X*, Symposium on Probability Methods in Analysis, Lecture Notes in Mathematics 31, Berlin 1967, p. 299-328.

[16] — *Sur la structure des lois indéfiniment divisibles dans les espaces vectoriels*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 11 (1969), p. 311-326.

[17] K. Urbanik, *Lévy's probability measures on Banach spaces*, Studia Mathematica 63 (1978), p. 283-308.

[18] Н. Н. Вахания, *Вероятностные распределения в линейных пространствах*, Тбилиси 1971.

[19] A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Paris 1940.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF WROCŁAW

Reçu par la Rédaction le 18. 11. 1977