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# ON THE EXISTENCE OF EXPONENTIAL MOMENTS OF RADEMACHER SUMS

BY

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Let $r_1, r_2, \ldots$ be a Rademacher sequence, i.e. $r_j$ are independent identically distributed random variables: $P(r_j = 1) = P(r_j = -1) = \frac{1}{2}$. It is well known (cf., for instance, [2], Chapter V, Section 8) that if $a_1, a_2, \ldots$ is a sequence of positive numbers such that $\sum a_j^2 < \infty$ (this assumption guarantees the convergence a.s. of $\sum a_j r_j$), then $\mathrm{E}\exp\left(t(\sum a_j r_j)^2\right) < \infty$ for every real $t$. In the present paper the existence of exponential moments of order greater than 2 of sums of Rademacher series will be examined. The question is when the inequality $\mathrm{E}\exp\left(t|\sum a_j r_j|^r\right) < \infty$ holds. We shall prove the following

THEOREM. *Let* $0 \leqslant q < 1$. *If* $a_1, a_2, \ldots$ *satisfies the condition*

$$\sum_{j=n+1}^{\infty} a_j^2 = O(n^{-q}),$$

*then for* $r < 2/(1-q)$ *the mean value* $\mathrm{E}\exp(t|\sum_{j=1}^{\infty} a_j r_j|^r)$ *is finite for every* $t$.

The Theorem is the generalization of Marcus' result in [1] and the most important part of our proof is also due to him. The special case $a_j = j^{-s}$ is considered in [1]. Our Theorem is also the strengthening of the following theorem proved by Hoffmann-Jørgensen in his unpublished paper:

*If* $q$ *is one of the numbers* $\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ *and* $a_1, a_2, \ldots$ *satisfies the condition*

$$\sum_{j=n+1}^{\infty} a_j^2 = o(n^{-q}),$$

*then* $\mathrm{E}\exp\left(t|\sum_{j=1}^{\infty} a_j r_j|^{2(1+q)}\right)$ *is finite for every* $t$.

Evidently, $2/(1-q) > 2(q+1)$ for $0 < q < 1$ and our Theorem is stronger.

From now on we assume that $a_1, a_2, \ldots$ is a sequence such that

$$a_j > 0 \quad \text{and} \quad \sum_{j=1}^{\infty} a_j^2 < \infty.$$

We shall use the following notation:

$$s_n = \sum_{j=1}^{n} a_j, \quad \sigma_n = \sum_{j=n+1}^{\infty} a_j^2, \quad \sigma = \sum_{j=1}^{\infty} a_j^2.$$

Let us begin with the following three lemmas:

LEMMA 1. $\mathrm{E}\exp(t\sum a_j r_j) \leqslant \exp(t^2\sigma/2)$.

LEMMA 2. $P(\sum a_j r_j > t) \leqslant \exp(-t^2/2\sigma)$.

Lemma 1 is a well-known fact and Lemma 2 is its immediate consequence.

LEMMA 3. $P(\sum a_j r_j > 2s_n) \leqslant \exp(-s_n^2/2\sigma_n)$.

Proof. We have

$$P(\sum_{j=1}^{\infty} a_j r_j > 2s_n) \leqslant P(\sum_{j=1}^{n} a_j r_j > s_n) + P(\sum_{j=n+1}^{\infty} a_j r_j > s_n).$$

The first term is equal to 0 and so Lemma 2 (applied to the sequence $r_{n+1}, r_{n+2}, \ldots$ instead of $r_1, r_2, \ldots$) gives us the required inequality.

Now, let us notice that to prove our Theorem it is sufficient to show that its assumption implies the estimate $\sigma_n = O(s_n^{-r})$ for every $r < 2q/(1-q)$ for $n$ large enough. Indeed, assume that the last estimate is valid, i.e. $\sigma_n \leqslant As_n^{-r}$ for some positive constant $A$ and for large $n$. Without loss of generality we may assume $\sum a_j = \infty$. For every $u$ sufficiently large there exists $n$ such that $u-1 < 2s_n < u$. By Lemma 3 we have

$$P(\sum a_j r_j > u) \leqslant P(\sum a_j r_j > 2s_n) \leqslant \exp(-s_n^2/2\sigma_n)$$

and

$$\exp(-s_n^2/2\sigma_n) \leqslant \exp(-(2A)^{-1}s_n^{2+r}) \leqslant \exp(-(2A)^{-1}[(u-1)/2]^{2+r}) \leqslant \exp(-Bu^{2+r})$$

Evidently, the obtained tail probability estimate implies

$$\mathrm{E}\exp(t|\sum_{j=1}^{\infty} a_j r_j|^{2+r}) < \infty.$$

Now, it is sufficient to notice that if $r$ runs over the interval $(0, 2q/(1-q))$, then $r+2$ runs over $(2, 2/(1-q))$.

Proof of the Theorem. Let us consider in the series $a_1+a_2+\ldots$ the following groups of terms:

$$a_1+a_2+(a_3+a_4)+(a_5+\ldots+a_8)+(a_9+\ldots+a_{16})+\ldots$$

The Schwarz inequality applied to the $(p+2)$-nd group gives

$$a_{2^p+1}+\dots+a_{2^{p+1}} \leqslant (a_{2^p+1}^2+\dots+a_{2^{p+1}}^2)^{1/2}\cdot 2^{p/2}.$$

Now, using the assumption $\sigma_n \leqslant An^{-q}$, we obtain

$$a_{2^p+1}+\dots+a_{2^{p+1}} \leqslant (A\cdot 2^{-pq}\cdot 2^p)^{1/2} = A^{1/2}\cdot 2^{p(1-q)/2}.$$

Hence

$$s_{2^{p+1}} \leqslant a_1 + A^{1/2}\sum_{k=1}^{p} 2^{k(1-q)/2} \leqslant D\cdot 2^{(1-q)(p+1)/2}$$

for some constant $D > 0$ (for instance, $D = a_1 + A^{1/2}(2^{(1-q)/2}-1)^{-1}$).

Let $2^p < n \leqslant 2^{p+1}$. We have

$$\sigma_n \leqslant \sigma_{2^p} \leqslant A\cdot 2^{-pq} \leqslant Cs_{2^{p+1}}^{[-2q(1-q)][p(p+1)]}, \quad \text{where } C > AD^{2pq(1-q)(p+1)}.$$

Given $r < 2q/(1-q)$ it is enough to choose $\hat{p}$ so large that

$$\frac{2q}{1-q}\,\frac{\hat{p}}{\hat{p}+1} > r.$$

This allows us to obtain $\sigma_n \leqslant Cs_n^{-r}$ for $2^p < n \leqslant 2^{p+1}$, where $p = \hat{p}$, $\hat{p}+1$, $\hat{p}+2, \dots$, i.e. for every $n > 2^{\hat{p}}$. Thus the proof is complete.

Remark 1. The exponent $2/(1-q)$, which appears in our Theorem, is the best possible.

In fact, Marcus [1] shows that although the sequence $a_j = j^{-(1-q)/2}$ satisfies the condition $\sigma_n = O(n^{-q})$, we have $\mathrm{E}\exp|\sum a_j r_j|^r = \infty$ for every $r > 2/(1-q)$.

Remark 2. If the condition $\sigma_n = O(n^{-1})$ is satisfied, then the random variable $\sum a_j r_j$ has finite exponential moments of all orders.

Remark 3. If the condition $\sigma_n = O(n^{-q})$ for some $q > 1$ is satisfied, then $\sum a_j < \infty$ and $\sum a_j r_j$ is a bounded random variable.

To prove this let us consider the following inequality which appears in the proof of the Theorem:

$$s_{2^{p+1}} \leqslant a_1 + A^{1/2}\sum_{k=1}^{p} 2^{k(1-q)/2}.$$

This is also true in the case $q > 1$ and the proof needs no change. If $q > 1$, then the geometric series on the right-hand side of this inequality converges, and hence the sequence of the partial sums $s_{2^p}$ of $\sum a_j$ is bounded.

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