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A FURTHER NOTE ON A CLASS OF I_0-SETS

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1. Introduction. A subset E of the real numbers R is an I_0-set [3] provided every bounded complex-valued function on E can be extended to an almost periodic function on R. Suppose that $\Lambda = \{q_j: j \in Z^+\}$ is a lacunary subset of R^+, that is, $q = \inf\{q_{j+1}/q_j: j \in Z^+\} > 1$. It is a classical result [6], [7] that every such set Λ is an I_0-set. In [2], another class of I_0-sets called blocked sets was investigated. These sets are constructed by summing pairs of elements from a lacunary set Λ in the following manner. Let $K = \{k_j\}$ be any subset of Λ, and let $\Lambda(k_j)$ be any sequence of disjoint subsets of Λ. Define the blocked set $E = \bigcup(k_j + \Lambda(k_j))$, $j \in Z^+$. If $K \cap \Lambda(k_j) = \emptyset$ for all j, then E is called a *restricted blocked set*. The following theorems were obtained in [2].

THEOREM 1. *Let $\Lambda = \{q_j\}$ be a lacunary set with lacunary ratio $q > 2$. Then any blocked set E formed from Λ is an I_0-set.*

THEOREM 2. *Let $\Lambda = \{q_j\}$ be a lacunary set with lacunary ratio $q > (1+\sqrt{5})/2$. In addition, suppose there exists a $v > 0$ such that $|2-(q_{j+1}/q_j)| \geqslant v$ for all j. Then any restricted blocked set E formed from Λ is an I_0-set.*

In this investigation examples are given to show that Theorem 1 is sharp and that the lacunary ratio in Theorem 2 cannot be lowered. Moreover, the techniques used to prove Theorems 1 and 2 are applied to obtain the next result.

THEOREM 3. *If E is any blocked set formed from any lacunary set $\Lambda \subset Z^+$, then E is a Sidon set.*

2. Examples. If q is sufficiently large, then Theorems 1 and 2 show that blocked or restricted blocked sets formed from Λ are I_0. On the other hand, if the lacunary ratio of Λ is too small, then there may exist blocked or restricted blocked sets formed from Λ which contain too many arithmetic relations to be I_0-sets. To show this we will need the following difference set criterion. Although a well-known topological proof using the Bohr compactification could be given, we prefer to use a different technique.

PROPOSITION 1. *Let $E \subset R$. Suppose there is a partition $E = E_0 \cup E_1$ and an infinite set $S \subset R$ such that $(S-S)\setminus\{0\} \subset (E_0-E_1)\cup(E_1-E_0)$. Then E is not an I_0-set.*

Proof. Let f be the function on E which is 0 on E_0 and 1 on E_1, and let g be any extension of f to R. If $s, t \in S$ and $s \neq t$, then either $s-t = m_0 - m_1$ or $s-t = m_1 - m_0$, where $m_0 \in E_0$ and $m_1 \in E_1$. For the sake of argument, let us assume that $s-t = m_0 - m_1$. Then

$$\|g_s - g_t\|_\infty = \|g - g_{t-s}\|_\infty \geqslant |g(m_1) - g(m_1 - t + s)| = |g(m_1) - g(m_0)| = 1.$$

From this it follows that the norm balls of radius 1/2 centered about g_s and g_t, respectively, are disjoint subsets of $L^\infty(R)$. Therefore the set of all translates of g is not a totally bounded subset of $L^\infty(R)$, and so g is not almost periodic.

Note that the hypothesis of Proposition 1 can be relaxed. For example, the existence of an inifnite sequence of finite subsets $S_n \subset R$ with $\operatorname{card}(S_n) \geqslant n$ and $(S_n - S_n)\setminus\{0\} \subset (E_0-E_1)\cup(E_1-E_0)$ is sufficient to show that E is not an I_0-set. Also Proposition 1 makes it easy to see that the union of two I_0-sets, e.g. $\{2^j\colon j \in Z^+\}$ and $\{2^j + j\colon j \in Z^+\}$, need not be an I_0-set [4, p. 132].

Example 1. There is a lacunary set $\Lambda \subset Z^+$ with lacunary ratio 2 and a blocked set formed from Λ which is not an I_0-set.

Set $n_0 = 1$; define $n_{2j-1} = 2^{3j-2}$ and $n_{2j} = 2^{3j-1} + j - 1$ for $j \in Z^+$. If $\Lambda = \{n_k\colon k = 0, 1, 2, \ldots\}$, then it is clear that Λ has lacunary ratio 2. For $j \in Z^+$, define $m_{2j-1} = n_{2j-1} + n_{2j-1} = 2^{3j-1}$ and $m_{2j} = n_{2j} + n_0 = 2^{3j-1} + j$, and consider the blocked set $E = \{m_k\colon k \in Z^+\}$ formed from Λ. Note that E is the union of $E_0 = \{m_{2j}\}$ and $E_1 = \{m_{2j-1}\}$. Since $(Z-Z)\setminus\{0\} \subset (E_0 - E_1)\cup(E_1 - E_0)$, Proposition 1 implies that E is not an I_0-set.

Example 2. There is a lacunary set Λ with lacunary ratio $(1+\sqrt{5})/2$ and a restricted blocked set formed from Λ which is not an I_0-set.

Write $q = (1+\sqrt{5})/2$ and set $n_0 = 1$. For $j \in Z^+$ define $n_{3j-2} = q^{4j-3}$, $n_{3j-1} = q^{4j-2}$, $n_{3j} = q^{4j-1} + j - 1$. If $\Lambda = \{n_k\colon k = 0, 1, 2, \ldots\}$, then Λ is clearly lacunary with lacunary ratio q. For $j \in Z^+$ define $m_{2j-1} = n_{3j-1} + n_{3j-2}$ and $m_{2j} = n_{3j} + n_0$. Note that $q^2 - q - 1 = 0$ implies $m_{2j} - m_{2j-1} = j$. Consider the blocked set $E = \{m_k\}$ formed from Λ. Observe that E is the union of $E_0 = \{m_{2j}\}$ and $E_1 = \{m_{2j-1}\}$ and that $(Z-Z)\setminus\{0\} \subset (E_0 - E_1)\cup(E_1 - E_0)$. Therefore E is not an I_0-set by Proposition 1.

3. Proof of Theorem 3. If E is a blocked set formed from a lacunary set $\Lambda \subset Z^+$ with lacunary ratio $q > 1$, then we will show that E is the union of a finite number of I_0-sets. Theorem 3 will then follow since I_0-subsets of Z are Sidon [5], and the union of a finite number of Sidon sets is again Sidon [1].

Define $v = 1/4$, $\delta = 1/120$, and $\varepsilon^{-1} = \mu = 300$. Choose $r \in Z^+$ large enough so that $q^r > \varepsilon^{-2}$ and $q > 1+q^{-r/2}$. Write $E = N \cup P \cup Q$ where

$$N = \{x \in E\colon\ x = k+l \text{ where } k \in K,\ l \in \Lambda(k), \text{ and } l/k < \varepsilon\},$$

$$P = \{x \in E\colon\ x = k+l \text{ where } k \in K,\ l \in \Lambda(k), \text{ and } \varepsilon \leqslant l/k \leqslant q^{r/2}\},$$

$$Q = \{x \in E\colon\ x = k+l \text{ where } k \in K,\ l \in \Lambda(k), \text{ and } q^{r/2} < l/k\}.$$

Using the notation in the paragraph following the proof of Lemma 9 in [2], we note that N is the union of the $2r^2$ sets $N(i, j, \sigma)$ where $1 \leqslant i, j \leqslant r$, $\sigma = \pm 1$. The proof of Lemma 22 in [2] with the above values of v, δ, and ε shows that each $N(i, j, \sigma)$ is an I_0-set. Next we claim that Q is a lacunary set. To see this, suppose that $k_1+l_1,\ k_2+l_2 \in Q$ where $k_i \in K$, $l_i \in \Lambda(k_i)$, and $q^{r/2} < l_i/k_i$ for $i = 1, 2$. By the definition of a blocked set, $l_1 \neq l_2$. Therefore, by reindexing if necessary, we may assume that $l_1 < l_2$. Then $(k_2+l_2)/(k_1+l_1) > (l_2/l_1)/[1+(k_1/l_1)] \geqslant q/(1+q^{-r/2})$, and so Q is lacunary with lacunary ratio $q/(1+q^{-r/2}) > 1$. To finish it is enough to show that P is a finite union of lacunary sets. If $K = \{k_j\}$ where $k_{j+1}/k_j \geqslant q$, then define $P(k_j) = \{x \in P\colon\ x = k_j+l \text{ where } l \in \Lambda(k_j)\}$. Since there are at most r elements $l \in \Lambda$ which satisfy $\varepsilon \leqslant l/k_j \leqslant q^{r/2}$, it follows that $\operatorname{card}(P(k_j)) \leqslant r$. For each $1 \leqslant i \leqslant r$ define $P(i) = \bigcup P(k_{jr+i})$, $j \in Z^+ \cup \{0\}$. Observe that if $x = k_{jr+i}+l \in P(k_{jr+i})$ and $x' = k_{(j+1)r+i}+l' \in P(k_{(j+1)r+i})$, then

$$x'/x \geqslant [(1+\varepsilon)k_{(j+1)r+i}]/[(1+q^{r/2})k_{jr+i}] \geqslant q^r(1+\varepsilon)/(1+q^{r/2}) > q^{r/2}.$$

Hence if we form a subset X of $P(i)$ by selecting at most one element from each $P(k_{jr+i})$, then X is lacunary with lacunary ratio at least $q^{r/2}$. But $P(i)$ can be written as the union of r such subsets X; consequently $P = \bigcup P(i)$, $1 \leqslant i \leqslant r$, is the union of r^2 lacunary sets.

Blocked sets in Theorem 3 were shown to be Sidon by proving that each is a finite union of I_0-sets. This suggests the following sequence of questions which the author was unable to answer.

(1) Does there exist a Sidon set which is not a finite union of I_0-sets **(P 1322)**? All known examples of Sidon sets in Z are finite unions of Rider sets. (Recall that $\{n_j\}$ is a *Rider set* provided for some $B > 0$ and for every positive integer s the number of representations of zero of the form $0 = \sum_{k=1}^{s} \varepsilon_k n_{j_k}$ ($\varepsilon_k = \pm 1$, $j_1 < \ldots < j_s$) does not exceed B^s.) Therefore (1) naturally leads to the next question.

(2) Is each Rider set in Z an I_0-set, or at least a finite union of I_0-sets **(P 1323)**? The last question is a special case of (1).

(3) Is each dissociate set in Z an I_0-set (**P 1324**)? (Recall that $\{n_j\}$ is *dissociate* provided that $\sum_{k=1}^{m} \varepsilon_k n_k = 0$ ($\varepsilon_k = 0, \pm 1, \pm 2$) implies that $\varepsilon_k = 0$ for $1 \leqslant k \leqslant m$.)

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